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Small-on-Large Geometric Anelasticity

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In this paper we are concerned with finding exact solutions for the stress fields of nonlinear solids with non-symmetric distributions of defects (or more generally finite eigenstrains) that are small perturbations of symmetric distributions of defects with known exact solutions. In the language of geometric mechanics this corresponds to finding a deformation that is a result of a perturbation of the metric of the Riemannian material manifold. We present a general framework that can be used for a systematic analysis of this class of anelasticity problems. This geometric formulation can be thought of as a material analogue of the classical *small-on-large* theory in nonlinear elasticity. We use the present small-on-large anelasticity theory to find exact solutions for the stress fields of some non-symmetric distributions of screw dislocations in incompressible isotropic solids.

1. Introduction

Mechanics of residually-stressed solids has been of interest to many researchers in solid mechanics for quite some time. In an anelastic deformation any measure of strain has a non-elastic component. This means that a non-vanishing strain does not necessarily correspond to a non-vanishing (conjugate) stress; only the elastic part of strain—the elastic strain—enters the constitutive equations. The remaining part of strain is called pre-strain or *eigenstrain* as coined by Mura [1]. One source of anelasticity is defects. Line defects in solids were mathematically introduced by Vito Volterra more than a century ago [2]. Volterra realized that such defects, which he called *distortions*, induce a self-equilibrated state of

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residual stresses. His calculations were done in the setting of linear elasticity. He introduced six types of line defects, three of which are now called dislocations (translational defects), and the other three are called disclinations (rotational defects). Other examples of anelasticity sources include non-uniform temperature distributions [3, 4, 5], bulk growth [6, 7, 8, 9, 10], accretion (surface growth) [11, 12], and swelling [13, 14, 15]. Following the pioneering works of Eckart [16] and Kondo [17], the multiplicative decomposition of deformation gradient was proposed by Bilby et al. [18] and Kröner [19] and has been extensively used in the literature to solve anelasticity problems (see [20, 21] for a further discussion on the origins and the use of the multiplicative decomposition in the mechanics literature). Alternatively, rather than using the conceptually ambiguous intermediate configuration in the framework of the multiplicative decomposition (cf. [4, 9] for detailed discussions), eigenstrains can be modeled using an abstract manifold (material manifold) that is possibly non-Euclidean [22, 23].

Evolution of defects in solids is an important and difficult problem when strains are finite. The complexity of the equations of anelasticity leaves little hope for finding exact solutions. A handful of exact solutions have been found using semi-inverse methods assuming some symmetric classes of deformations (these are all somewhat related to Ericksen's universal deformations [24]). In the case of defects, examples can be seen for dislocations and disclinations in [25, 26, 27, 28, 29, 30], and for point defects and discombinations in [31, 32, 33]. The existing exact solutions correspond to highly symmetric distributions of defects. As soon as this symmetry is broken, the governing equations start to be utterly complicated leaving no choice but for numerical computations. One possibility for extending the class of problems amenable to exact solutions is to study those defect distributions that are perturbations of the highly symmetric ones. This is what we call small-on-large anelasticity in this paper, which is a material analogue of the small-on-large theory of Green et al. [34] (further discussion and several applications of this theory can be found in [35, 36]). Given a distribution of some source of anelasticity with a known exact solution, we perturb the distribution and solve for the induced small elastic deformations. This is achieved by linearizing the governing equations about the known solution with respect to the perturbation. Even in the case when one fails to find exact solutions in this framework, the linearized governing equations are much easier to solve numerically. In this paper we are concerned with the change of the state of stress (residual stress) of a hyperelastic body with a given distribution of defects, or more generally a source of anelasticity, under a perturbation of the defect distribution. A change of the defect distribution changes the geometry of the material manifold, and consequently changes the metric of the underlying Riemannian material manifold. Such calculations have two immediate applications: i) Suppose one has an analytic solution for the stress field of a given distribution of defects (dislocations, disclinations, point defects, or a combination of them—discombinations [33]). Can one calculate the residual stress field of the body if the defect distribution is perturbed slightly? ii) One may be interested in stability of a defect distribution. If the defect distribution is allowed to perturb, would the total energy of the system change? Any reduction of the energy of the system may indicate instability of the defect distribution.

This paper is organized as follows. In §2, we briefly review the basic concepts of Riemannian geometry and geometric elasticity needed in our formulation of small-on-large anelasticity. In §3, we formulate the governing equations for the small deformations induced by a perturbation of the distribution of finite eigenstrains. In our geometric framework, such a perturbation is equivalent to perturbing the material metric. In §4, we solve several examples of screw dislocations that are perturbations of an axi-symmetric distribution of screw dislocations in an infinite body made of an incompressible isotropic solid. Conclusions are given in §5.

2. An overview of nonlinear elasticity

We briefly review in the following some elements of the geometric formulation of nonlinear elasticity and anelasticity. For more details, see for example [29, 37]. Let B be a three-dimensional body identified with a three-dimensional Riemannian manifold $(\mathcal{B}, \boldsymbol{G})$[1]—the material manifold where the body is stress-free. Let $(\mathcal{S}, \boldsymbol{g})$ be a Riemannian ambient space manifold, which we assume is Euclidean, i.e., $\mathcal{S} = \mathbb{R}^3$ and $\boldsymbol{g}$ its usual Euclidean metric.[2] We adopt the standard convention to denote objects and indices by uppercase characters in the material manifold $\mathcal{B}$ (e.g., $X \in \mathcal{B}$) and by lowercase characters in the spatial manifold $\mathcal{S}$ (e.g., $x \in \mathcal{S}$). We denote by $\{X^A\}$ and $\{x^a\}$ the local coordinate charts on $\mathcal{B}$ and $\mathcal{S}$, respectively, by $\partial_A = \frac{\partial}{\partial X^A}$ and $\partial_a = \frac{\partial}{\partial x^a}$, we denote the corresponding local coordinate bases, respectively, and by $\{dX^A\}$ and $\{dx^a\}$, we denote the corresponding dual bases. We also adopt Einstein's repeated index summation convention, e.g., $u^i v_i := \sum_i u^i v_i$. Let $\nabla^{\boldsymbol{G}}$, and $\nabla^{\boldsymbol{g}}$ be the Levi-Civita connections of $(\mathcal{B}, \boldsymbol{G})$, and $(\mathcal{S}, \boldsymbol{g})$, repectively. We denote their respective Christoffel symbols by $\Gamma^A{}_{BC}$, and $\gamma^a{}_{bc}$, in the local coordinate charts $\{X^A\}$ and $\{x^a\}$, respectively. By a configuration of $\mathcal{B}$, we mean a smooth embedding $\varphi : \mathcal{B} \to \mathcal{S}$. We denote the set of all configurations of $\mathcal{B}$ by $\mathcal{C}$. A motion of $\mathcal{B}$ is a smooth curve in $\mathcal{C}$, i.e., a mapping $t \in \mathbb{R}^+ \to \varphi_t \in \mathcal{C}$. We introduce the notations $\varphi(X,t) := \varphi_X(t) := \varphi_t(X)$.

The deformation gradient $\boldsymbol{F}$ is defined as the tangent map of $\varphi_t : \mathcal{B} \to \mathcal{S}$, i.e., $\boldsymbol{F}(X,t) := T\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{S}$. We denote the transpose of $\boldsymbol{F}$ by $\boldsymbol{F}^{\mathsf{T}}$ and it is defined such that $\forall (\boldsymbol{W}, \boldsymbol{w}) \in (T_X\mathcal{B} \times T_{\varphi_t(X)}\mathcal{S}) : \boldsymbol{g}(\boldsymbol{F}\boldsymbol{W}, \boldsymbol{w}) = \boldsymbol{G}(\boldsymbol{W}, \boldsymbol{F}^{\mathsf{T}}\boldsymbol{w})$. In components, $(F^{\mathsf{T}})^A{}_a = g_{ab} F^b{}_B G^{AB}$. The Jacobian J relates the material and spatial Riemannian volume elements $dV(X, \boldsymbol{G})$ and $dv(x, \boldsymbol{g})$ by $dv(\varphi_t(X), \boldsymbol{g}) = J(X, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}) dV(X, \boldsymbol{G})$. It can be shown that $J = \sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}}} \det \boldsymbol{F}$. The right Cauchy-Green deformation tensor is defined as $\boldsymbol{C} = \boldsymbol{F}^{\mathsf{T}}\boldsymbol{F}$. In components, $C^A{}_B = G^{AK} F^a{}_K F^b{}_B g_{ab}$. Note that $\boldsymbol{C}^\flat$ agrees with the pull-back of the spatial metric $\boldsymbol{g}$ by φ, i.e., $\boldsymbol{C}^\flat = \varphi^* \boldsymbol{g}$, where $(.)^\flat$ denotes the flat operator for lowering tensor indices. The left Cauchy-Green deformation tensor (also called Finger tensor) is defined as $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^{\mathsf{T}}$. In components, $b^a{}_b = F^a{}_A F^c{}_B G^{AB} g_{cb}$. Note that $\boldsymbol{b}^{-\flat}$ agrees with the push-forward of the material metric $\boldsymbol{G}$ by φ, i.e., $\boldsymbol{b}^{-\flat} = \varphi_* \boldsymbol{G}$, where $(.)^{-\flat}$ denotes the inverse operator followed by the flat operator. We define the convective manifold as the Riemannian manifold $(\mathcal{B}, \boldsymbol{C}^\flat)$. Let $\nabla^{\boldsymbol{C}}$ be the Levi-Civita connection of $(\mathcal{B}, \boldsymbol{C}^\flat)$. We denote its corresponding Christoffel symbols in the local coordinate chart $\{X^A\}$ by $\tilde{\Gamma}^A{}_{BC}$.

The material velocity of the motion is defined as the mapping $\boldsymbol{V} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S}$ such that $\boldsymbol{V}(X,t) := \varphi_{X*}\partial_t \in T_{\varphi_X(t)}\mathcal{S}$, which in components reads $V^a(X,t) = \frac{\partial \varphi^a}{\partial t}(X,t)$. The spatial velocity is defined as the mapping $\boldsymbol{v} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S}$ such that $\boldsymbol{v}(x,t) := \boldsymbol{V}(\varphi_t^{-1}(x), t) \in T_x\mathcal{S}$. The material acceleration is defined as the mapping $\boldsymbol{A} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S}$ such that $\boldsymbol{A}(X,t) := D_t^{\boldsymbol{g}} \boldsymbol{V}(X,t) \in T_{\varphi(X)}\mathcal{S}$, where $D_t^{\boldsymbol{g}}$ denotes the covariant derivative along φ_X. In components, $A^a = \frac{\partial V^a}{\partial t} + \gamma^a{}_{bc} V^b V^c$. The spatial acceleration is defined as the mapping $\boldsymbol{a} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S}$ such that $\boldsymbol{a}(x,t) := \boldsymbol{A}(\varphi_t^{-1}(x), t) \in T_x\mathcal{S}$. In components, $a^a = \frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^b} v^b + \gamma^a{}_{bc} v^b v^c$.

We denote the material and spatial mass densities by ρ_o and ρ, respectively. The conservation of mass in local form reads $\rho J = \rho_o$, which is equivalent to

$$\frac{d\rho}{dt} + \rho \, \mathrm{div}_{\boldsymbol{g}} \boldsymbol{v} = 0 \,,$$

where $\mathrm{div}_{\boldsymbol{g}}$ denotes the spatial divergence operator.

We assume that the body is made of a hyperelastic material, so that the constitutive model is given by an energy function $\mathcal{W} = \tilde{\mathcal{W}}(X, \boldsymbol{F}, \boldsymbol{g}, \boldsymbol{G})$[3] per unit undeformed volume, and the Cauchy stress tensor is given by [40]

$$\boldsymbol{\sigma} = \frac{2}{J} \frac{\partial \tilde{\mathcal{W}}}{\partial \boldsymbol{g}} \,, \tag{2.1}$$

[1]The material manifold need not be Riemannian, e.g., dislocations can be modeled by torsion [29, 38], and point defects by non-metricity [32]. Note, however, that only the underlying Riemannian metric is needed to calculate (residual) stresses.

[2]See [39] for an example of a non-Euclidean ambient space.

[3]The dependence of the energy function $\tilde{\mathcal{W}}$ on the metrics follows from the fact that $\tilde{\mathcal{W}}$ is a scalar that depends on the deformation gradient $\boldsymbol{F}$. This requires the metrics to obtain a scalar out of it, e.g., $\mathrm{tr}(\boldsymbol{F}^{\mathsf{T}}\boldsymbol{F}) = F^a{}_A F^b{}_B G^{AB} g_{ab}$.

which in components reads $\sigma^{ab} = \frac{2}{J}\frac{\partial \tilde{\mathcal{W}}}{\partial g_{ab}}$. We can alternatively consider $\mathcal{W} = \hat{\mathcal{W}}(X, \boldsymbol{C}^\flat, \boldsymbol{G})$ and the convected stress tensor $\boldsymbol{\Sigma} = \varphi_t^* \boldsymbol{\sigma}$ is written as [41]

$$\boldsymbol{\Sigma} = \frac{2}{J}\frac{\partial \hat{\mathcal{W}}}{\partial \boldsymbol{C}^\flat}\,, \tag{2.2}$$

which in components reads $\Sigma^{ab} = \frac{2}{J}\frac{\partial \hat{\mathcal{W}}}{\partial C_{AB}}$. If the material is incompressible, we have $J = 1$ and the stress tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\Sigma}$ are written as

$$\boldsymbol{\sigma} = 2\frac{\partial \tilde{\mathcal{W}}}{\partial \boldsymbol{g}} - p\,\boldsymbol{g}^\sharp\,, \quad \boldsymbol{\Sigma} = 2\frac{\partial \hat{\mathcal{W}}}{\partial \boldsymbol{C}^\flat} - p\,\boldsymbol{C}^{-\sharp}\,, \tag{2.3}$$

where p is the Lagrange multiplier associated with the incompressibility constraint, and $(.)^{-\sharp}$ denotes the inverse operator followed by the sharp operator for raising tensor indices. If the material is isotropic, the strain-energy function is expressed as a function of the principal invariants $I_1 = \mathrm{tr}\boldsymbol{C}$, $I_2 = \frac{1}{2}(\mathrm{tr}(\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2))$, and J , i.e., $\mathcal{W} = \bar{\mathcal{W}}(X, I_1, I_2, J)$, and the stress tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\Sigma}$ can be written as [36, 40]

$$\boldsymbol{\sigma} = \left(\bar{\mathcal{W}}_J + \frac{2I_2}{J}\bar{\mathcal{W}}_{I_2}\right)\boldsymbol{g}^\sharp + \frac{2}{J}\bar{\mathcal{W}}_{I_1}\boldsymbol{b}^\sharp - 2J\bar{\mathcal{W}}_{I_2}\boldsymbol{b}^{-\sharp}\,, \tag{2.4a}$$

$$\boldsymbol{\Sigma} = \frac{2}{J}\left(\bar{\mathcal{W}}_{I_1} + I_1\bar{\mathcal{W}}_{I_2}\right)\boldsymbol{G}^\sharp - \frac{2}{J}\bar{\mathcal{W}}_{I_2}\boldsymbol{C}^\sharp + \bar{\mathcal{W}}_J\boldsymbol{C}^{-\sharp}\,, \tag{2.4b}$$

where $\bar{\mathcal{W}}_{I_1} = \frac{\partial \bar{\mathcal{W}}}{\partial I_1}$, $\bar{\mathcal{W}}_{I_2} = \frac{\partial \bar{\mathcal{W}}}{\partial I_2}$, and $\bar{\mathcal{W}}_J = \frac{\partial \bar{\mathcal{W}}}{\partial J}$. If the material is incompressible and isotropic, one has

$$\boldsymbol{\sigma} = \left(2I_2\bar{\mathcal{W}}_{I_2} - p\right)\boldsymbol{g}^\sharp + 2\bar{\mathcal{W}}_{I_1}\boldsymbol{b}^\sharp - 2\bar{\mathcal{W}}_{I_2}\boldsymbol{b}^{-\sharp}\,, \tag{2.5a}$$

$$\boldsymbol{\Sigma} = 2\left(\bar{\mathcal{W}}_{I_1} + I_1\bar{\mathcal{W}}_{I_2}\right)\boldsymbol{G}^\sharp - 2\bar{\mathcal{W}}_{I_2}\boldsymbol{C}^\sharp - p\boldsymbol{C}^{-\sharp}\,. \tag{2.5b}$$

In spatial form, the balance of linear and angular momenta read

$$\mathrm{div}_{\boldsymbol{g}}\boldsymbol{\sigma} + \rho\mathbf{f} = \rho\boldsymbol{a}\,, \quad \boldsymbol{\sigma}^{\mathsf{T}} = \boldsymbol{\sigma}\,, \tag{2.6}$$

where $\mathbf{f}$ denotes the body force per unit mass. The balance of linear and angular momenta in terms of the convected stress tensor read [41] (Note that, since $\nabla^{\boldsymbol{C}} = \varphi_t^*\nabla^{\boldsymbol{G}}$, the convective balance of momenta (2.7) can alternatively be obtained directly from the classical spatial balance of momenta (2.6).)

$$\mathrm{Div}_{\boldsymbol{C}}\boldsymbol{\Sigma} + \rho\varphi_t^*\mathbf{F} = \rho\varphi_t^*\boldsymbol{A}\,, \quad \boldsymbol{S}^{\mathsf{T}} = \boldsymbol{S}\,, \tag{2.7}$$

where $\mathrm{Div}_{\boldsymbol{C}}$ denotes the divergence operator with respect to $\boldsymbol{C}^\flat$, and $\mathbf{F} := \mathbf{f}\circ\varphi_t$.

3. Small-on-Large Deformations Due to a Material Metric Perturbation

In this section, we formulate a theory of small superposed deformations due to a perturbation of the material metric. Given a motion φ_t with respect to a reference configuration $(\mathcal{B}, \boldsymbol{G})$, we consider a 1-parameter family of metrics $\boldsymbol{G}_\epsilon$ such that $\boldsymbol{G}_0 = \boldsymbol{G}$. We want to understand how the state of stress in the body is affected by such a perturbation. Note that a perturbation of the material metric is due to a perturbation of the source of anelasticity, e.g. a defect density. The variation of the material metric is defined as

$$\delta\boldsymbol{G} := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\boldsymbol{G}_\epsilon\,.$$

For a small enough ϵ, one can write $\boldsymbol{G}_\epsilon = \boldsymbol{G} + \epsilon\delta\boldsymbol{G} + \boldsymbol{o}(\epsilon)$. Note that even though the deformation is seemingly independent of the material metric, changing the material metric may affect the equilibrium configuration of the body at any given time t . Hence a perturbation of the material metric may lead to a perturbation $\varphi_{t,\epsilon}$ of the motion, such that $\varphi_{t,0} = \varphi_t$ is the equilibrium configuration corresponding to the metric $\boldsymbol{G}_0=\boldsymbol{G}$. We define its corresponding variation as

$$\delta\varphi_t(X):=\varphi_{t,X*}\partial_\epsilon\big|_{\epsilon=0}\in T_{\varphi_t(X)}\mathcal{S}\,,$$

that is, $\delta\varphi_t=\delta\varphi_t^a\partial_a$ and $\delta\varphi_t^a(X):=\frac{d\varphi_{X,t}}{d\epsilon}|_{\epsilon=0}$. Note that $\delta\varphi\circ\varphi^{-1}$ is the displacement field in the classical theory of linear elasticity and we denote it by $\boldsymbol{U}=\delta\varphi\circ\varphi^{-1}$. Since $\mathcal{S}=\mathbb{R}^3$, using the linear structure of $\mathbb{R}^3$, one can write for a small enough ϵ: $\varphi_\epsilon=\varphi+\epsilon\delta\varphi+o(\epsilon)$. Given the configuration φ resulting in the stress field $\boldsymbol{\sigma}$, the perturbed configuration φ_ϵ due to the material metric perturbation $\boldsymbol{G}_\epsilon$ induces a stress field, which for a small enough ϵ reads $\boldsymbol{\sigma}_\epsilon=\boldsymbol{\sigma}+\epsilon\delta\boldsymbol{\sigma}+o(\epsilon)$. In the following, we formulate the governing equations to solve for $\delta\varphi$ and find $\delta\boldsymbol{\sigma}$ is terms of $\delta\boldsymbol{G}$ and $\delta\varphi$.

As ϵ varies, for fixed X and t, the right Cauchy-Green tensor $\boldsymbol{C}_\epsilon^\flat$ remains in the same space $\mathcal{T}^2(T_X^*\mathcal{B})$, the set of $\binom{0}{2}$-rank tensors at X. Thus, it makes sense to define its variation as $\delta\boldsymbol{C}^\flat=\left.\frac{d\boldsymbol{C}_\epsilon^\flat}{d\epsilon}\right|_{\epsilon=0}$. One can write $\delta\boldsymbol{C}^\flat$ as follows

$$\delta\boldsymbol{C}^\flat=\left.\frac{d}{d\epsilon}\boldsymbol{C}_\epsilon^\flat\right|_{\epsilon=0}=\varphi_t^*\left.\frac{d}{d\epsilon}\left[\varphi_{t*}\varphi_{t,\epsilon}^*\boldsymbol{g}\right]\right|_{\epsilon=0}=\varphi_t^*\boldsymbol{L}_{\boldsymbol{U}}\boldsymbol{g}=\varphi_t^*\left(\nabla^{\boldsymbol{g}}\boldsymbol{U}^\flat+\left[\nabla^{\boldsymbol{g}}\boldsymbol{U}^\flat\right]^{\mathsf{T}}\right)=2\varphi_t^*\boldsymbol{\epsilon}\,,$$

where $(.)^{\mathsf{T}}$ denotes the transpose operator, and $\boldsymbol{\epsilon}=\frac{1}{2}\left(\nabla^{\boldsymbol{g}}\boldsymbol{U}^\flat+\left[\nabla^{\boldsymbol{g}}\boldsymbol{U}^\flat\right]^{\mathsf{T}}\right)$ is the linearized strain. The variation of the Jacobian of the motion reads[4]

$$\delta J=\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\sqrt{\frac{\det\boldsymbol{C}_\epsilon^\flat}{\det\boldsymbol{G}}}=\left(\boldsymbol{\epsilon}{:}\boldsymbol{g}^\sharp-\frac{1}{2}\delta\boldsymbol{G}{:}\boldsymbol{G}^\sharp\right)J\,,\tag{3.1}$$

where ":" denotes the double contraction tensor product. Using $\rho J=\rho_o$ and the above equation (3.1), the variation of the spatial mass density reads

$$\delta\rho=-\left(\boldsymbol{\epsilon}{:}\boldsymbol{g}^\sharp-\frac{1}{2}\delta\boldsymbol{G}{:}\boldsymbol{G}^\sharp\right)\rho\,.\tag{3.2}$$

Note that when ϵ varies, the terms in the balance of linear momentum (2.7) are vectors that remain in the same vector space $T_X\mathcal{B}$.[5] Hence, one can write its variation as

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left[\operatorname{Div}_{\boldsymbol{C}_\epsilon}\boldsymbol{\Sigma}_\epsilon+\rho_\epsilon\varphi_{\epsilon,t}^*\boldsymbol{B}\right]=\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left[\rho_\epsilon\varphi_{\epsilon,t}^*\boldsymbol{A}_\epsilon\right]\,,$$

which, by expanding the divergence term in local coordinates, transforms to read

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left[\left(\Sigma_\epsilon^{AB}{}_{,B}+\Sigma_\epsilon^{AK}\tilde{\Gamma}_\epsilon^B{}_{BK}+\Sigma_\epsilon^{BK}\tilde{\Gamma}_\epsilon^A{}_{BK}\right)\partial_A+\rho_\epsilon\varphi_{\epsilon,t}^*\boldsymbol{B}\right]=\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left[\rho_\epsilon\varphi_{\epsilon,t}^*\boldsymbol{A}_\epsilon\right]\,.\tag{3.3}$$

For different values of ϵ and fixed X and t, $\boldsymbol{\Sigma}_\epsilon$ lie in the same space $\mathcal{T}^2(T_X\mathcal{B})$. Hence, one can define $\delta\boldsymbol{\Sigma}=\left.\frac{d\boldsymbol{\Sigma}_\epsilon}{d\epsilon}\right|_{\epsilon=0}$, which is computed in (3.4*a*) following (2.2). On the other hand, the variation of the Cauchy stress can be defined as the push-forward of that of the convected stress, i.e., $\delta\boldsymbol{\sigma}=\varphi_{t*}\delta\boldsymbol{\Sigma}$. Therefore, one finds

$$\delta\boldsymbol{\Sigma}=\frac{4}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial\boldsymbol{C}^\flat\partial\boldsymbol{C}^\flat}{:}\varphi_t^*\boldsymbol{\epsilon}+\frac{2}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial\boldsymbol{G}\partial\boldsymbol{C}^\flat}{:}\delta\boldsymbol{G}-\left(\boldsymbol{\epsilon}{:}\boldsymbol{g}^\sharp-\frac{1}{2}\delta\boldsymbol{G}{:}\boldsymbol{G}^\sharp\right)\boldsymbol{\Sigma}\,,\tag{3.4a}$$

$$\delta\boldsymbol{\sigma}=\frac{4}{J}\frac{\partial^2\tilde{\mathcal{W}}}{\partial\boldsymbol{g}\partial\boldsymbol{g}}{:}\boldsymbol{\epsilon}+\frac{2}{J}\frac{\partial^2\tilde{\mathcal{W}}}{\partial\boldsymbol{G}\partial\boldsymbol{g}}{:}\delta\boldsymbol{G}-\left(\boldsymbol{\epsilon}{:}\boldsymbol{g}^\sharp-\frac{1}{2}\delta\boldsymbol{G}{:}\boldsymbol{G}^\sharp\right)\boldsymbol{\sigma}\,.\tag{3.4b}$$

We define the following fourth order elasticity tensors:

$$\mathbb{C}:=\frac{4}{J}\frac{\partial^2\tilde{\mathcal{W}}}{\partial\boldsymbol{g}\partial\boldsymbol{g}}\,,\qquad\mathbb{D}:=\frac{2}{J}\frac{\partial^2\tilde{\mathcal{W}}}{\partial\boldsymbol{G}\partial\boldsymbol{g}}\,,\tag{3.5}$$

[4]Recall that if $\det\boldsymbol{A}\neq0$, one has $\frac{d\det\boldsymbol{A}}{d\boldsymbol{A}}=(\det\boldsymbol{A})\,\boldsymbol{A}^{-\mathsf{T}}$. Here, $\det\boldsymbol{G}\neq0$ and $\det\boldsymbol{C}^\flat\neq0$.

[5]However, note that when ϵ varies, the terms in the balance of linear momentum (2.6) are vectors that lie in the vector space $T_{\varphi_{t,\epsilon}(X)}\mathcal{S}$, in which the base point $\varphi_{t,\epsilon}(X)$ depends on ϵ.

which in components read $\mathbb{C}^{abcd} = \frac{4}{J}\frac{\partial^2 \tilde{\mathcal{W}}}{\partial g_{ab}\partial g_{cd}}$, and $\mathbb{D}^{abAB} = \frac{2}{J}\frac{\partial^2 \tilde{\mathcal{W}}}{\partial G_{AB}\partial g_{ab}}$. Using (2.7), (3.2) and (3.4*a*), the governing equation (3.3) for the incremental stress transforms to[6]

$$\begin{aligned}
&\mathrm{Div}_{\boldsymbol{C}}\left(\frac{4}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial \boldsymbol{C}^\flat \partial \boldsymbol{C}^\flat}:\varphi_t^*\boldsymbol{\epsilon} + \frac{2}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial \boldsymbol{G}\partial \boldsymbol{C}^\flat}:\delta\boldsymbol{G}\right) - d_{\mathcal{B}}\left(\boldsymbol{\epsilon}:\boldsymbol{g}^\sharp - \frac{1}{2}\delta\boldsymbol{G}:\boldsymbol{G}^\sharp\right)\cdot\boldsymbol{\Sigma}\\
&-2\Sigma^{BK}C^{-AL}\ \varphi_t^*\boldsymbol{\epsilon}|_{LM}\,\tilde{\Gamma}^M{}_{BK}\partial_A + \Sigma^{BK}C^{-AL}\Big[\ \varphi_t^*\boldsymbol{\epsilon}|_{BL,K} + \varphi_t^*\boldsymbol{\epsilon}|_{KL,B} - \varphi_t^*\boldsymbol{\epsilon}|_{BK,L}\Big]\partial_A\\
&-2\Sigma^{AK}C^{-BL}\ \varphi_t^*\boldsymbol{\epsilon}|_{LM}\,\tilde{\Gamma}^M{}_{BK}\partial_A + \Sigma^{AK}C^{-BL}\Big[\ \varphi_t^*\boldsymbol{\epsilon}|_{BL,K} + \varphi_t^*\boldsymbol{\epsilon}|_{KL,B} - \varphi_t^*\boldsymbol{\epsilon}|_{BK,L}\Big]\partial_A\\
&\qquad\qquad +\rho\,\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{B}\right]\Big|_{\epsilon=0} = \rho\,\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{A}_\epsilon\right]\Big|_{\epsilon=0}\ ,
\end{aligned} \tag{3.6}$$

where $d_{\mathcal{B}}$ denotes the exterior derivative operator on $\mathcal{B}$, i.e., for a function $f:\mathcal{B}\to\mathbb{R}$, one has $d_{\mathcal{B}}f = \frac{\partial f}{\partial X^A}dX^A$. Denoting by a double stroke $(.)_{||}$ the convective covariant derivative, i.e., the covariant derivative in the convective manifold $(\mathcal{B},\boldsymbol{C})$, one can write $(\varphi_t^*\boldsymbol{\epsilon})_{BL,K} + (\varphi_t^*\boldsymbol{\epsilon})_{KL,B} - (\varphi_t^*\boldsymbol{\epsilon})_{BK,L} = (\varphi_t^*\boldsymbol{\epsilon})_{BL||K} + (\varphi_t^*\boldsymbol{\epsilon})_{KL||B} - (\varphi_t^*\boldsymbol{\epsilon})_{BK||L} + 2\,(\varphi_t^*\boldsymbol{\epsilon})_{LM}\,\tilde{\Gamma}^M{}_{BK}$. One can also show that

$$\begin{aligned}
&C^{-BL}\left[(\varphi_t^*\boldsymbol{\epsilon})_{BL||K} + (\varphi_t^*\boldsymbol{\epsilon})_{KL||B} - (\varphi_t^*\boldsymbol{\epsilon})_{BK||L}\right]\\
&= \left[C^{-BL}\,(\varphi_t^*\boldsymbol{\epsilon})_{BL||K} + C^{-BL}\,(\varphi_t^*\boldsymbol{\epsilon})_{KL||B} - C^{-BL}\,(\varphi_t^*\boldsymbol{\epsilon})_{BK||L}\right]\\
&= \left[\left(\boldsymbol{C}^{-1}:\varphi_t^*\boldsymbol{\epsilon}\right)_{,K} + C^{-IJ}\,(\varphi_t^*\boldsymbol{\epsilon})_{KJ||I} - C^{-JI}\,(\varphi_t^*\boldsymbol{\epsilon})_{JK||I}\right] = \left(\boldsymbol{g}^\sharp:\boldsymbol{\epsilon}\right)_{,K}\ .
\end{aligned}$$

On the other hand, one has $d_{\mathcal{B}}\left(\boldsymbol{\epsilon}:\boldsymbol{g}^\sharp\right)\cdot\boldsymbol{\Sigma} = \left(\boldsymbol{g}^\sharp:\boldsymbol{\epsilon}\right)_{,K}\Sigma^{KA}\partial_A$. Therefore, (3.6) is simplified to read

$$\begin{aligned}
&\mathrm{Div}_{\boldsymbol{C}}\left(\frac{4}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial \boldsymbol{C}^\flat \partial \boldsymbol{C}^\flat}:\varphi_t^*\boldsymbol{\epsilon} + \frac{2}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial \boldsymbol{G}\partial \boldsymbol{C}^\flat}:\delta\boldsymbol{G}\right) + d_{\mathcal{B}}\left(\frac{1}{2}\delta\boldsymbol{G}:\boldsymbol{G}^\sharp\right)\cdot\boldsymbol{\Sigma}\\
&+\Sigma^{BK}C^{-AL}\Big[\varphi_t^*\boldsymbol{\epsilon}|_{BL||K} + \varphi_t^*\boldsymbol{\epsilon}|_{KL||B} - \varphi_t^*\boldsymbol{\epsilon}|_{BK||L}\Big]\partial_A + \rho\,\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{B}\right]\Big|_{\epsilon=0} = \rho\,\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{A}_\epsilon\right]\Big|_{\epsilon=0}\ .
\end{aligned} \tag{3.7}$$

Recall that, $\nabla^{\boldsymbol{C}} = \varphi_t^*\nabla^{\boldsymbol{g}}$. Thus, one can write

$$(\varphi_t^*\boldsymbol{\epsilon})_{AB||C} = F^a{}_A F^b{}_B F^c{}_C\,\epsilon_{ab|c} = \frac{1}{2}F^a{}_A F^b{}_B F^c{}_C\left(U_{a|bc} + U_{b|ac}\right)\ .$$

Assuming that the ambient space is flat, it follows that $U_{a|bc} = U_{a|cb}$. Hence, it is straightforward to show that $(\varphi_t^*\boldsymbol{\epsilon})_{BL||K} + (\varphi_t^*\boldsymbol{\epsilon})_{KL||B} - (\varphi_t^*\boldsymbol{\epsilon})_{BK||L} = F^b{}_B F^k{}_K F^l{}_L U_{l|bk}$. For the acceleration vector, one has

$$\begin{aligned}
\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{A}_\epsilon\right]\Big|_{\epsilon=0} &= \varphi_t^*\boldsymbol{L}_{\boldsymbol{U}}\boldsymbol{A} = \varphi_t^*\left[\frac{\partial A_\epsilon^a}{\partial\epsilon}\Big|_{\epsilon=0}\partial_a + \nabla^{\boldsymbol{g}}_{\boldsymbol{U}}\boldsymbol{A} - \nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right] = \varphi_t^*\left[D^{\boldsymbol{g}}_\epsilon\boldsymbol{A} - \nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right]\\
&= \varphi_t^*\left[D^{\boldsymbol{g}}_\epsilon D^{\boldsymbol{g}}_t\boldsymbol{V} - \nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right] = \varphi_t^*\left[D^{\boldsymbol{g}}_t D^{\boldsymbol{g}}_\epsilon\boldsymbol{V} + \nabla^{\boldsymbol{g}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{V} - \nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right]\\
&= \varphi_t^*\left[D^{\boldsymbol{g}}_t D^{\boldsymbol{g}}_t\boldsymbol{U} + \nabla^{\boldsymbol{g}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{V} - \nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right]\ ,
\end{aligned}$$

where $D^{\boldsymbol{g}}_\epsilon$ denotes the covariant derivative along $\epsilon\to\varphi_{\epsilon,t}(X)$, for X and t fixed, and where we used $D^{\boldsymbol{g}}_\epsilon D^{\boldsymbol{g}}_t\boldsymbol{V} = D^{\boldsymbol{g}}_t D^{\boldsymbol{g}}_\epsilon\boldsymbol{V} + \nabla^{\boldsymbol{g}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{V}$, since we assume a flat ambient space. We also use the symmetry lemma [42] to write $D^{\boldsymbol{g}}_\epsilon\boldsymbol{V} = D^{\boldsymbol{g}}_t\boldsymbol{U}$. For the body force vector, one similarly has $\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{B}\right]\Big|_{\epsilon=0} = \varphi_t^*\boldsymbol{L}_{\boldsymbol{U}}\boldsymbol{B} = \varphi_t^*\left[\nabla^{\boldsymbol{g}}_{\boldsymbol{U}}\boldsymbol{B} - \nabla^{\boldsymbol{g}}_{\boldsymbol{B}}\boldsymbol{U}\right]$. Finally, using the above results and pushing

[6]Recall that the Christoffel symbols for the convective Levi-Civita connection, i.e., the Levi-Civita connection for the convective manifold $\left(\mathcal{B},\boldsymbol{C}^\flat\right)$, read $\tilde{\Gamma}^A{}_{BK} = \frac{1}{2}C^{-AL}\left(C_{BL,K} + C_{KL,B} + C_{BK,L}\right)$.

forward (3.7) by φ_t, one obtains the following balance of linear momentum for the perturbed motion

$$\begin{aligned}&\mathrm{div}_{\boldsymbol{g}}\left(\mathbb{C}{:}\boldsymbol{\epsilon}+\mathbb{D}{:}\delta\boldsymbol{G}\right)+\varphi_{t*}d_{\mathcal{B}}\left(\frac{1}{2}\delta\boldsymbol{G}{:}\boldsymbol{G}^\sharp\right)\cdot\boldsymbol{\sigma}\\&\qquad+\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}{:}\boldsymbol{\sigma}+\rho\left(\nabla^{\boldsymbol{g}}_{\boldsymbol{U}}\boldsymbol{B}-\nabla^{\boldsymbol{g}}_{\boldsymbol{B}}\boldsymbol{U}\right)=\rho\left(D^{\boldsymbol{g}}_tD^{\boldsymbol{g}}_t\boldsymbol{U}+\nabla^{\boldsymbol{g}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{V}-\nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right),\end{aligned}\tag{3.8}$$

where $\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}{:}\boldsymbol{\sigma}=\sigma^{ab}\nabla^{\boldsymbol{g}}_{\partial_a}\nabla^{\boldsymbol{g}}_{\partial_b}\boldsymbol{U}=\sigma^{ab}U^c{}_{|ba}\partial_c$. If the material is incompressible, the variation of the convected and the Cauchy stress tensors are written as

$$\delta\boldsymbol{\Sigma}=\varphi_t^*\left(\mathbb{C}{:}\boldsymbol{\epsilon}+\mathbb{D}{:}\delta\right)-\delta p\,\boldsymbol{C}^{-\sharp}+2p\,\varphi_t^*\boldsymbol{\epsilon}^\sharp\,,\tag{3.9a}$$

$$\delta\boldsymbol{\sigma}=\mathbb{C}{:}\boldsymbol{\epsilon}+\mathbb{D}{:}\delta\boldsymbol{G}-\delta p\,\boldsymbol{g}^\sharp+2p\boldsymbol{\epsilon}^\sharp,\tag{3.9b}$$

where $\delta p=\frac{d}{d\epsilon}\Big|_{\epsilon=0}p_\epsilon$ is the resulting pressure variation, which can also be interpreted as the Lagrange multiplier associated with the constraint $\delta J=0$. Therefore, for an incompressible solid, the balance of linear momentum for the perturbed motion reads

$$\begin{aligned}&\mathrm{div}_{\boldsymbol{g}}\delta\boldsymbol{\sigma}+\varphi_{t*}d_{\mathcal{B}}\left(\frac{1}{2}\delta\boldsymbol{G}{:}\boldsymbol{G}^\sharp\right)\cdot\boldsymbol{\sigma}+\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}{:}\boldsymbol{\sigma}+\rho\left(\nabla^{\boldsymbol{g}}_{\boldsymbol{U}}\boldsymbol{B}-\nabla^{\boldsymbol{g}}_{\boldsymbol{B}}\boldsymbol{U}\right)\\&\qquad\qquad=\rho\left(D^{\boldsymbol{g}}_tD^{\boldsymbol{g}}_t\boldsymbol{U}+\nabla^{\boldsymbol{g}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{V}-\nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right).\end{aligned}\tag{3.10}$$

Remark 3.1. *Note that for an isotropic solid, one can show that the components of the elasticity tensors* (3.5) *read*

$$\begin{aligned}\mathbb{C}^{abcd}=&\left(\bar{W}_J+J\bar{W}_{JJ}+4I_2\bar{W}_{I_2J}+\frac{4I_2}{J}\bar{W}_{I_2}+\frac{4I_2^2}{J}\bar{W}_{I_2I_2}\right)g^{ab}g^{cd}+\frac{4}{J}\bar{W}_{I_1I_1}b^{ab}b^{cd}\\&-\left(\bar{W}_J+\frac{2I_2}{J}\bar{W}_{I_2}\right)\left(g^{ac}g^{bd}+g^{ad}g^{bc}\right)+\left(2\bar{W}_{I_1J}+\frac{4I_2}{J}\bar{W}_{I_1I_2}\right)\left(g^{ab}b^{cd}+b^{ab}g^{cd}\right)\\&-J\left(2J\bar{W}_{I_2J}+4\bar{W}_{I_2}+4I_2\bar{W}_{I_2I_2}\right)\left(g^{ab}b^{-cd}+b^{-ab}g^{cd}\right)+4J^3\bar{W}_{I_2I_2}b^{-ab}b^{-cd}\\&+2J\bar{W}_{I_2}\left(b^{-ac}g^{bd}+b^{-ad}g^{bc}+b^{-bc}g^{ad}+b^{-bd}g^{ac}\right)-4J\bar{W}_{I_1I_2}\left(b^{-ab}b^{cd}+b^{ab}b^{-cd}\right),\end{aligned}\tag{3.11a}$$

$$\begin{aligned}\mathbb{D}^{abAB}=&-\left(\frac{1}{2}\bar{W}_J+\frac{J}{2}\bar{W}_{JJ}+2I_2\bar{W}_{I_2J}+\frac{2I_2}{J}\bar{W}_{I_2}+\frac{2I_2^2}{J}\bar{W}_{I_2I_2}\right)g^{ab}G^{AB}\\&-\frac{1}{J}\bar{W}_{I_1}\left(F^a{}_KF^b{}_LG^{AK}G^{BL}+F^b{}_KF^a{}_LG^{AK}G^{BL}\right)-2J^3\bar{W}_{I_2I_2}b^{-ab}C^{-AB}\\&+J\left(J\bar{W}_{I_2J}+2\bar{W}_{I_2}+2I_2\bar{W}_{I_2I_2}\right)\left(b^{-ab}G^{AB}+g^{ab}C^{-AB}\right)-\frac{2}{J}\bar{W}_{I_1I_1}b^{ab}C^{AB}\\&+2J\bar{W}_{I_1I_2}\left(b^{-ab}C^{AB}+b^{ab}C^{-AB}\right)-\left(\bar{W}_{I_1J}+\frac{2I_2}{J}\bar{W}_{I_1I_2}\right)\left(b^{ab}G^{AB}+g^{ab}C^{AB}\right)\\&-J\bar{W}_{I_2}\left(g^{ak}g^{bl}F^{-A}{}_kF^{-B}{}_l+g^{bk}g^{al}F^{-A}{}_kF^{-B}{}_l\right).\end{aligned}\tag{3.11b}$$

For an incompressible isotropic solid, the components of the elasticity tensors can be obtained from (3.11) *by setting $J=1$ and removing the terms containing $\bar{W}_J$.*

4. Examples of Material Metric Perturbations in an Infinitely Long Cylindrical Bar with an Axi-Symmetric Distribution of Parallel Screw Dislocations

In this section, we solve examples of perturbed dislocation distributions. Starting from a dislocation distribution with an existing equilibrium solution, we perturb it and solve for the induced small elastic deformations due to the resulting material metric perturbation. We consider the example of a cylindrically-symmetric distribution of parallel screw dislocations in a cylinder made of an incompressible, isotropic, and radially inhomogeneous nonlinear elastic solid, i.e., a solid with an energy function that can be written as $\mathcal{W} = \bar{\mathcal{W}}(R, I_1, I_2)$. Using the geometric theory of nonlinear dislocation mechanics introduced in [29], we first construct the stress-free Weitzenböck material manifold for an arbitrary cylindrically-symmetric parallel screw-dislocations distribution. Next, considering a perturbation of the axi-symmetric dislocation distribution following §3, we solve for the induced small elastic deformations and the corresponding stress field.

(a) Material metric perturbation

In a cylindrical coordinate system (R, Θ, Z), we consider a distribution of cylindrically-symmetric screw dislocations parallel to the Z-axis by assuming a Z-oriented radially-symmetric Burgers' vector density $b = b(R)$. Let us consider a perturbation of this Burgers' vector distribution, i.e., we take a one-parameter family of Burgers' vectors $b_\epsilon(R, \Theta, Z)$ such that $b_0(R, \Theta, Z) = b(R)$. We define its variation as $\delta b = \frac{d}{d\epsilon} b_\epsilon \Big|_{\epsilon=0}$. The given distribution of Burgers' vectors is equivalent to having the following torsion 2-forms

$$\mathcal{T}^1 = \mathcal{T}^2 = 0\,, \quad \mathcal{T}^3_\epsilon = \frac{b_\epsilon(R, \Theta, Z)}{2\pi} \vartheta^1 \wedge \vartheta^2\,.$$

Following the method of Cartan's moving frames [43], we look for an orthonormal coframe field of the form $\vartheta^1 = dR$, $\vartheta^2 = R d\Theta$, $\vartheta^3 = dZ + f_\epsilon(R, \Theta, Z) d\Theta$, for some function $f_\epsilon = f_\epsilon(R, \Theta, Z)$ to be determined. Denoting by $\omega^\alpha{}_\beta$ the connection 1-forms, Cartan's first structural equations, $\mathcal{T}^\alpha = d\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta$, for $\alpha = 1, 2, 3$, give one the following non-zero connection coefficients

$$\omega^1{}_{22} = -\frac{1}{R}\,, \quad \omega^1{}_{32} = -\frac{1}{2}\left(\frac{f_{\epsilon,R}}{R} - \frac{b_\epsilon}{2\pi}\right)\,, \quad \omega^2{}_{13} = \omega^3{}_{21} = \frac{1}{2}\left(\frac{f_{\epsilon,R}}{R} - \frac{b_\epsilon}{2\pi}\right)\,, \quad \omega^2{}_{33} = \frac{f_{\epsilon,Z}}{R}\,.$$

Hence, the connection 1-forms read

$$\omega^1{}_2 = -\frac{1}{R}\vartheta^2 - \frac{1}{2}\left(\frac{f_{\epsilon,R}}{R} - \frac{b_\epsilon}{2\pi}\right)\vartheta^3,\ \omega^2{}_3 = \frac{1}{2}\left(\frac{f_{\epsilon,R}}{R} - \frac{b_\epsilon}{2\pi}\right)\vartheta^1 + \frac{f_{\epsilon,Z}}{R}\vartheta^3,\ \omega^3{}_1 = \frac{1}{2}\left(\frac{f_{\epsilon,R}}{R} - \frac{b_\epsilon}{2\pi}\right)\vartheta^2.$$

Cartan's second structural equations, $\mathcal{R}^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge^\gamma{}_\beta$, for $\alpha, \beta = 1, 2, 3$, along with the flatness of the material manifold yield,[7] $f_{\epsilon,R} = R\frac{b_\epsilon}{2\pi}$, $f_{\epsilon,Z} = 0$. Therefore, $b_{\epsilon,Z} = 0$, and hence $b_\epsilon = b_\epsilon(R, \Theta)$, i.e., a Z-dependent Burgers' vector cannot be accommodated using the assumed coframe field. It then follows that $f_\epsilon(R, \Theta) = \frac{1}{2\pi}\int_0^R \xi b_\epsilon(\xi, \Theta) d\xi$, and the perturbed material metric in the coordinate frame is written as

$$\boldsymbol{G}_\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 + f_\epsilon^2(R, \Theta) & f_\epsilon(R, \Theta) \\ 0 & f_\epsilon(R, \Theta) & 1 \end{pmatrix}.$$

Hence, the variation of the material metric is written as

$$\delta\boldsymbol{G} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2f(R)\delta f(R, \Theta) & \delta f(R, \Theta) \\ 0 & \delta f(R, \Theta) & 0 \end{pmatrix},$$

where

$$f(R) = \frac{1}{2\pi}\int_0^R \xi b(\xi) d\xi \qquad \text{and} \qquad \delta f(R, \Theta) = \frac{1}{2\pi}\int_0^R \xi \delta b(\xi, \Theta) d\xi\,.$$

[7]For dislocations, the material manifold is by construction a Weitzenböck manifold, i.e., it is flat and has a compatible connection with a possibly non-zero torsion [38].

Knowing that $b_0 = b(R)$, we have $f_0 = f(R) = \frac{1}{2\pi}\int_0^R \xi b(\xi) d\xi$ and $\boldsymbol{G}_0 = \boldsymbol{G}(R)$ is the metric for the axi-symmetric parallel screw dislocations

$$\boldsymbol{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 + f^2(R) & f(R) \\ 0 & f(R) & 1 \end{pmatrix}.$$

Note that $\operatorname{tr}(\delta\boldsymbol{G}) = \delta\boldsymbol{G}{:}\boldsymbol{G}^\sharp = 0$.

(b) Stress perturbation

Let us first find the residual stress field for the finite axi-symmetric distribution assuming an incompressible isotropic solid. Based on the symmetry of the problem, we look for an embedding of the material manifold in the Euclidean ambient space such that, in cylindrical coordinates (r, θ, z), we have $\varphi(R, \Theta, Z) = (r(R), \Theta, Z)$. Then, the deformation gradient reads $\boldsymbol{F} = \operatorname{diag}(r'(R), 1, 1)$ and the Jacobian is written as $J = rr'/R$. Using the incompressibility condition, i.e., $J = 1$, and assuming that $r(0) = 0$ to fix the rigid body translation of the body, we find that $r(R) = R$. Hence, the standard Euclidean metric for $\mathcal{S} = \mathbb{R}^3$ in cylindrical coordinates (r, θ, z) reads $\boldsymbol{g} = \operatorname{diag}(1, R^2, 1)$ and the only non-zero Christoffel symbols are $\gamma^r{}_{\theta\theta} = -R$ and $\gamma^\theta{}_{r\theta} = \frac{1}{R}$. The Finger deformation tensor is written as

$$\boldsymbol{b}^\sharp = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2} & -\frac{f(R)}{R^2} \\ 0 & -\frac{f(R)}{R^2} & 1 + \frac{f^2(R)}{R^2} \end{pmatrix}.$$

Following (2.5*a*) and denoting $\alpha(R) = 2\bar{\mathcal{W}}_{I_1}(R, I_1(R), I_2(R))$ and $\beta(R) = 2\bar{\mathcal{W}}_{I_2}(R, I_1(R), I_2(R))$, the non-zero Cauchy stress components read

$$\sigma^{rr} = -p(R, \Theta, Z) + \alpha(R) + \left(\frac{f^2(R)}{R^2} + 2\right)\beta(R), \quad \sigma^{\theta\theta} = \frac{1}{R^2}\left[-p(R, \Theta, Z) + \alpha(R) + 2\beta(R)\right],$$

$$\sigma^{zz} = -p(R, \Theta, Z) + \left(\frac{f^2(R)}{R^2} + 1\right)\alpha(R) + \left(\frac{f^2(R)}{R^2} + 2\right)\beta(R), \quad \sigma^{\theta z} = -\frac{f(R)}{R^2}\left[\alpha(R) + \beta(R)\right].$$

Note that $I_1(R) = I_2(R) = 3 + f^2(R)/R^2$. The θ and z-equilibrium equations imply that $p = p(R)$, and the radial equilibrium equation is simplified to read $\sigma^{rr}{}_{,R} + \frac{1}{R}\sigma^{rr} - R\sigma^{\theta\theta} = 0$. Assuming a traction-free boundary condition on the boundary of the cylinder at $R = R_o$, we solve the above equation for $p = p(R)$ and it follows that the non-zero Cauchy stress components are

$$\begin{aligned} \sigma^{rr} &= \int_R^{R_o} \frac{f^2(\xi)}{\xi^3}\beta(\xi)d\xi, \quad \sigma^{\theta\theta} = \frac{1}{R^2}\left[\int_R^{R_o} \frac{f^2(\xi)}{\xi^3}\beta(\xi)d\xi - \frac{f^2(R)}{R^2}\beta(R)\right], \\ \sigma^{zz} &= \int_R^{R_o} \frac{f^2(\xi)}{\xi^3}\beta(\xi)d\xi + \frac{f^2(R)}{R^2}\alpha(R), \quad \sigma^{\theta z} = -\frac{f(R)}{R^2}\left[\alpha(R) + \beta(R)\right]. \end{aligned} \tag{4.1}$$

Next we formulate the governing equations for superposed small elastic deformation and compute the incremental deformation and residual stresses due to the perturbation δb. In cylindrical coordinates (r, θ, z), we look for solutions of the form $\delta\varphi(R, \Theta) = \boldsymbol{U}(R, \Theta) = (\delta r(R, \Theta), \delta\theta(R, \Theta), \delta z(R, \Theta))$. Hence, $\nabla^{\boldsymbol{g}}\boldsymbol{U}$ reads

$$U^a{}_{|b} = \begin{pmatrix} \delta r_{,R} & \delta r_{,\Theta} - R\delta\theta & 0 \\ \delta\theta_{,R} + \frac{\delta\theta}{R} & \delta\theta_{,\Theta} + \frac{\delta r}{R} & 0 \\ \delta z_{,R} & \delta z_{,\Theta} & 0 \end{pmatrix}.$$

Recalling that the linearized strain reads $\boldsymbol{\epsilon}=\frac{1}{2}\left(\nabla^{\boldsymbol{g}}\boldsymbol{U}^{\flat}+\left[\nabla^{\boldsymbol{g}}\boldsymbol{U}^{\flat}\right]^{\mathsf{T}}\right)$, one can write

$$\boldsymbol{\epsilon}=\begin{pmatrix}\delta r_{,R} & \frac{1}{2}\left(\delta r_{,\Theta}+R^2\delta\theta_{,R}\right) & \frac{1}{2}\delta z_{,R}\\ \frac{1}{2}\left(\delta r_{,\Theta}+R^2\delta\theta_{,R}\right) & R^2\left(\delta\theta_{,\Theta}+\frac{1}{R}\delta r\right) & \frac{1}{2}\delta z_{,\Theta}\\ \frac{1}{2}\delta z_{,R} & \frac{1}{2}\delta z_{,\Theta} & 0\end{pmatrix}.$$

Note that $\delta\boldsymbol{G}{:}\boldsymbol{G}^{\sharp}=0$, and hence, the incompressibility condition $\delta J=0$ using (3.1) is simplified to read

$$\frac{1}{R}\left(R\delta r\right)_{,R}+\delta\theta_{,\Theta}=0\,. \tag{4.2}$$

In the absence of body forces, the equilibrium equation (3.10) simplifies to read

$$\operatorname{div}_{\boldsymbol{g}}\delta\boldsymbol{\sigma}+\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}{:}\boldsymbol{\sigma}=\mathbf{0}\,, \tag{4.3}$$

where we recall that $\delta\boldsymbol{\sigma}=\left(\mathbb{C}{:}\boldsymbol{\epsilon}+\mathbb{D}{:}\delta\boldsymbol{G}-\delta p\,\boldsymbol{g}^{\sharp}+2p\,\boldsymbol{\epsilon}^{\sharp}\right)$, $\delta p=\delta p(R,\Theta)$ is the Lagrange multiplier associated with the incompressibility condition $\delta J=0$ (4.2), and $p=p(R)$ is the Lagrange multiplier associated with the incompressibility condition $J=1$. Note that $\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}$ can be written in local coordinates as

$$U^a{}_{|bc}=\begin{pmatrix}\begin{pmatrix}\delta r_{,RR}\\ -R\delta\theta_{,R}-\frac{\delta r_{,\Theta}}{R}+\delta r_{,R\Theta}\\ 0\end{pmatrix} & \begin{pmatrix}-R\delta\theta_{,R}-\frac{\delta r_{,\Theta}}{R}+\delta r_{,R\Theta}\\ -2R\delta\theta_{,\Theta}+\delta r_{,\Theta\Theta}+R\delta r_{,R}-\delta r\\ 0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix}\\ \begin{pmatrix}\frac{2\delta\theta_{,R}}{R}+\delta\theta_{,RR}\\ \frac{R\delta r_{,R}-\delta r}{R^2}+\delta\theta_{,R\Theta}\\ 0\end{pmatrix} & \begin{pmatrix}\frac{R\delta r_{,R}-\delta r}{R^2}+\delta\theta_{,R\Theta}\\ \delta\theta_{,\Theta\Theta}+R\delta\theta_{,R}+\frac{2\delta r_{,\Theta}}{R}\\ 0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix}\\ \begin{pmatrix}\delta z_{,RR}\\ -\frac{\delta z_{,\Theta}}{R}+\delta z_{,R\Theta}\\ 0\end{pmatrix} & \begin{pmatrix}-\frac{\delta z_{,\Theta}}{R}+\delta z_{,R\Theta}\\ \delta z_{,\Theta\Theta}+R\delta z_{,R}\\ 0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix}\end{pmatrix}. \tag{4.4}$$

For the sake of simplifying the calculations, let us assume that the body is made of a generalized neo-Hookean solid, i.e., the energy function has the form $\mathcal{W}=\bar{\mathcal{W}}(I_1)$. Hence, it follows from (4.1) that the Cauchy stress reads

$$\boldsymbol{\sigma}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -2\frac{f(R)}{R^2}\bar{\mathcal{W}}_{I_1}\\ 0 & -2\frac{f(R)}{R^2}\bar{\mathcal{W}}_{I_1} & 2\frac{f^2(R)}{R^2}\bar{\mathcal{W}}_{I_1}\end{pmatrix}. \tag{4.5}$$

Thus, recalling that $\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}{:}\boldsymbol{\sigma}=U^a{}_{|bc}\sigma^{bc}\partial_a$, one finds from (4.4) and (4.5) that $\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}{:}\boldsymbol{\sigma}=\mathbf{0}$. Also, following (3.11), the elasticity tensors simplify to

$$\mathbb{C}{:}\boldsymbol{\epsilon}=4\bar{\mathcal{W}}_{I_1I_1}(\boldsymbol{b}^{\sharp}{:}\boldsymbol{\epsilon})\boldsymbol{b}^{\sharp}\,,\quad \mathbb{D}{:}\delta\boldsymbol{G}=-2\bar{\mathcal{W}}_{I_1I_1}(\boldsymbol{C}^{\sharp}{:}\delta\boldsymbol{G})\boldsymbol{b}^{\sharp}-2\bar{\mathcal{W}}_{I_1}\varphi_{t*}\delta\boldsymbol{G}^{\sharp}\,. \tag{4.6}$$

However, using the incompressibility condition (4.2), we have $\boldsymbol{b}^{\sharp}{:}\boldsymbol{\epsilon}=\frac{1}{R}\left(R\delta r\right)_{,R}+\delta\theta_{,\Theta}-\frac{f}{R^2}\delta z_{,\Theta}=-\frac{f}{R^2}\delta z_{,\Theta}$. Therefore

$$\mathbb{C}{:}\boldsymbol{\epsilon}=\frac{1}{R^4}\begin{pmatrix}-4R^2f\bar{\mathcal{W}}_{I_1I_1}\delta z_{,\Theta} & 0 & 0\\ 0 & -4f\bar{\mathcal{W}}_{I_1I_1}\delta z_{,\Theta} & 4f^2\bar{\mathcal{W}}_{I_1I_1}\delta z_{,\Theta}\\ 0 & 4f^2\bar{\mathcal{W}}_{I_1I_1}\delta z_{,\Theta} & -4f\left(R^2+f^2\right)\bar{\mathcal{W}}_{I_1I_1}\delta z_{,\Theta}\end{pmatrix}.$$

On the other hand $\boldsymbol{C}^{\sharp}{:}\delta\boldsymbol{G}=-2\frac{f}{R^2}\delta f$, and one can easily obtain

$$\mathbb{D}{:}\delta\boldsymbol{G}=\begin{pmatrix}\frac{4}{R^2}\bar{\mathcal{W}}_{I_1I_1}f\delta f & 0 & 0\\ 0 & \frac{4}{R^4}\bar{\mathcal{W}}_{I_1I_1}f\delta f & -2\left(\frac{1}{R^2}\bar{\mathcal{W}}_{I_1}+\frac{2f^2}{R^4}\bar{\mathcal{W}}_{I_1I_1}\right)\delta f\\ 0 & -2\left(\frac{1}{R^2}\bar{\mathcal{W}}_{I_1}+\frac{2f^2}{R^4}\bar{\mathcal{W}}_{I_1I_1}\right)\delta f & 4\left(\frac{1}{R^2}\bar{\mathcal{W}}_{I_1}+\frac{R^2+f^2}{R^4}\bar{\mathcal{W}}_{I_1I_1}\right)f\delta f\end{pmatrix}.$$

Therefore, the equilibrium equations (4.3) simplify to read

$$\mathrm{div}_{\boldsymbol{g}}\delta\boldsymbol{\sigma}=\mathbf{0}\,, \tag{4.7}$$

where

$$\begin{aligned}
\delta\sigma^{rr}&=\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^2}\left(\delta f-\delta z_{,\Theta}\right)+4\bar{\mathcal{W}}_{I_1}\delta r_{,R}-\delta p\,,\quad \delta\sigma^{r\theta}=2\bar{\mathcal{W}}_{I_1}\left(\frac{\delta r_{,\Theta}}{R^2}+\delta\theta_{,R}\right),\\
\delta\sigma^{\theta\theta}&=\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^4}\left(\delta f-\delta z_{,\Theta}\right)-\frac{1}{R^2}\left(\delta p+4\bar{\mathcal{W}}_{I_1}\delta r_{,R}\right),\\
\delta\sigma^{\theta z}&=-\frac{2\left(2f^2\bar{\mathcal{W}}_{I_1I_1}+R^2\bar{\mathcal{W}}_{I_1}\right)}{R^4}\left(\delta f-\delta z_{,\Theta}\right),\quad \delta\sigma^{rz}=2\bar{\mathcal{W}}_{I_1}\delta z_{,R}\,,\\
\delta\sigma^{zz}&=\frac{4f}{R^2}[(\bar{\mathcal{W}}_{I_1}+\bar{\mathcal{W}}_{I_1I_1})\delta f-\bar{\mathcal{W}}_{I_1I_1}\delta z_{,\Theta}]+\frac{4f^3\bar{\mathcal{W}}_{I_1I_1}}{R^4}\left(\delta f-\delta z_{,\Theta}\right)-\delta p\,.
\end{aligned} \tag{4.8}$$

Writing (4.7) in components along with the incompressibility condition (4.2) gives the following system of partial differential equations

$$\begin{aligned}
&\frac{\partial}{\partial R}\left[\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^2}\delta z_{,\Theta}-4\bar{\mathcal{W}}_{I_1}\delta r_{,R}+\delta p\right]\\
&-\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}\left[4R\delta r_{,R}+\delta r_{,\Theta\Theta}+R^2\delta\theta_{,R\Theta}\right]=\frac{\partial}{\partial R}\left[\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^2}\delta f\right],
\end{aligned} \tag{4.9a}$$

$$\begin{aligned}
&\frac{\partial}{\partial R}\left[\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}\left(\delta r_{,\Theta}+R^2\delta\theta_{,R}\right)\right]-\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^4}\delta z_{,\Theta\Theta}\\
&+\frac{2\bar{\mathcal{W}}_{I_1}}{R^3}\left[3\delta r_{,\Theta}-2R\delta r_{,R\Theta}+3R^2\delta\theta_{,R}\right]-\frac{1}{R^2}\delta p_{,\Theta}=-\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^4}\delta f_{,\Theta}\,,
\end{aligned} \tag{4.9b}$$

$$\begin{aligned}
&\frac{\partial}{\partial R}\left[2\bar{\mathcal{W}}_{I_1}\delta z_{,R}\right]+\frac{2\bar{\mathcal{W}}_{I_1}}{R}\delta z_{,R}\\
&+\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}+2R^2\bar{\mathcal{W}}_{I_1}}{R^4}\delta z_{,\Theta\Theta}=\left[\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}}{R^4}+\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}\right]\delta f_{,\Theta}\,,
\end{aligned} \tag{4.9c}$$

$$\delta r+R\delta r_{,R}+R\delta\theta_{,\Theta}=0\,. \tag{4.9d}$$

The boundary conditions corresponding to zero incremental boundary traction read $\delta\sigma^{rr}\left(R_o,\Theta\right)=0$, $\delta\sigma^{r\theta}\left(R_o,\Theta\right)=0$, and $\delta\sigma^{rz}\left(R_o,\Theta\right)=0$, which following (4.8) can be written as

$$\left[4\bar{\mathcal{W}}_{I_1}\delta r_{,R}-\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^2}\delta z_{,\Theta}-\delta p\right]_{(R_o,\Theta)}=-\left[\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^2}\delta f\right]_{(R_o,\Theta)}, \tag{4.10a}$$

$$\left[\frac{\delta r_{,\Theta}}{R^2}+\delta\theta_{,R}\right]_{(R_o,\Theta)}=0\,, \tag{4.10b}$$

$$\delta z_{,R}\left(R_o,\Theta\right)=0\,. \tag{4.10c}$$

In order to fix the rigid body motion of the cylinder, we assume that

$$\delta r(0,\Theta)=0\,,\quad \delta\theta(0,\Theta)=0\,,\quad \delta z(0,\Theta)=0\,. \tag{4.11}$$

Note that the continuity of the traction across any radial plane of constant Θ gives $\delta\sigma^{\theta z}\left(R,\Theta\right)=\delta\sigma^{\theta z}\left(R,\Theta+2\pi\right)$, $\delta\sigma^{\theta\theta}\left(R,\Theta\right)=\delta\sigma^{\theta\theta}\left(R,\Theta+2\pi\right)$, and $\delta\sigma^{\theta r}\left(R,\Theta\right)=\delta\sigma^{\theta r}\left(R,\Theta+2\pi\right)$. Also, in order to preserve the structural integrity of the cylinder, one must have $\delta r\left(R,\Theta\right)=\delta r\left(R,\Theta+2\pi\right)$, $\delta\theta\left(R,\Theta\right)=\delta\theta\left(R,\Theta+2\pi\right)$, and $\delta z\left(R,\Theta\right)=\delta z\left(R,\Theta+2\pi\right)$. Thus, it follows that δr, $\delta\theta$, δz, and δp are 2π-periodic functions with respect to Θ.

Note that $\delta z=\delta z(R,\Theta)$ can be obtained from (4.9c). Given the solution $\delta z=\delta z(R,\Theta)$ for (4.9c), we observe that the following functions are the unique solution for the system of linear ordinary differential equations (4.9) satisfying the boundary conditions (4.10) and (4.11):

$$\delta r=0\,,\quad \delta\theta=0\,,\quad \delta p=\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^2}\left(\delta f-\delta z_{,\Theta}\right)\,. \tag{4.12}$$

Therefore, following (4.8) and (4.12), the variation of the Cauchy stress tensor reads

$$\delta\boldsymbol{\sigma}=\begin{pmatrix} 0 & 0 & 2\bar{\mathcal{W}}_{I_1}\delta z_{,R} \\ 0 & 0 & -\left(\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}+\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}}{R^4}\right)(\delta f-\delta z_{,\Theta}) \\ 2\bar{\mathcal{W}}_{I_1}\delta z_{,R} & -\left(\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}+\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}}{R^4}\right)(\delta f-\delta z_{,\Theta}) & \frac{4f\bar{\mathcal{W}}_{I_1}}{R^2}\delta f+\frac{4f^3\bar{\mathcal{W}}_{I_1I_1}}{R^4}(\delta f-\delta z_{,\Theta}) \end{pmatrix}. \tag{4.13}$$

Let us first solve (4.9c) for $\delta z=\delta z(R,\Theta)$ to complete the solution (4.12). Recalling that δz is 2π-periodic with respect to Θ and assuming that δf is periodic as well, we can represent them by the following Fourier series

$$\delta z=\sum_{k=-\infty}^{\infty}\delta z_k(R)e^{\mathrm{i}k\Theta}\,,\quad \delta f=\sum_{k=-\infty}^{\infty}\delta f_k(R)e^{\mathrm{i}k\Theta}\,, \tag{4.14}$$

where $\mathrm{i}=\sqrt{-1}$, and for $k\in\mathbb{Z}$, δz_k and δf_k are the complex-valued Fourier coefficients given by

$$\delta z_k(R)=\frac{1}{2\pi}\int_0^{2\pi}\delta z(R,\zeta)e^{-\mathrm{i}k\zeta}d\zeta\,,\quad \delta f_k(R)=\frac{1}{2\pi}\int_0^{2\pi}\delta f(R,\zeta)e^{-\mathrm{i}k\zeta}d\zeta\,. \tag{4.15}$$

Substituting the Fourier series (4.14) into the partial differential equation (4.9c) for $k\in\mathbb{Z}$, we find

$$2\bar{\mathcal{W}}_{I_1}\delta z_k{}''+\left[2\frac{d\bar{\mathcal{W}}_{I_1}}{dR}+\frac{2\bar{\mathcal{W}}_{I_1}}{R}\right]\delta z_k{}'-\left[\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}}{R^4}+\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}\right]k^2\delta z_k=\left[\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}}{R^4}+\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}\right]\mathrm{i}k\delta f_k\,, \tag{4.16}$$

where $\delta z_k{}'=\frac{d\delta z_k}{dR}$, and $\delta z_k{}''=\frac{d^2\delta z_k}{dR^2}$. Note that δf_k can also be written as $\delta f_k=\frac{1}{2\pi}\int_0^R\xi\delta b_k(\xi)d\xi$, where δb_k is the k^{th} Fourier coefficient of δb. The boundary conditions for δz from (4.10) and (4.11) transform in terms of its Fourier coefficients to the following relations

$$\delta z_k(0)=0\,,\quad \delta z_k{}'(R_o)=0,\quad k\in\mathbb{Z}. \tag{4.17}$$

Therefore, we have transformed the real partial differential equation (4.9c) into a set of complex ordinary differential equations (4.16).

(c) Energy of a perturbed dislocation distribution

We next calculate the change in energy due to a small perturbation of the defect distribution to the first order in the defect perturbation. For a given distribution of screw dislocations, the energy per unit length in a cylinder made of a generalized neo-Hookean solid is written as

$$W=\int_0^{2\pi}\int_0^{R_o}\mathcal{W}(I_1(R,\Theta))RdRd\Theta\,.$$

Therefore, the variation of the energy following an arbitrary perturbation $\delta b=\delta b(R,\Theta)$ is written as

$$\delta W=\int_0^{2\pi}\int_0^{R_o}\left.\frac{d\mathcal{W}(I_{1\epsilon}(R,\Theta))}{d\epsilon}\right|_{\epsilon=0}RdRd\Theta=\int_0^{2\pi}\int_0^{R_o}\mathcal{W}_{I_1}(I_1(R,\Theta))\delta I_1(R,\Theta))RdRd\Theta\,.$$

Note that $\delta I_1=2\boldsymbol{\epsilon}{:}\boldsymbol{b}^\sharp+\delta\boldsymbol{G}{:}\boldsymbol{C}^\sharp=\frac{2f(R)}{R^2}\left[\delta f(R,\Theta)-\delta z_{,\Theta}(R,\Theta)\right]$. Therefore[8]

$$\delta W=\int_0^{R_o}\int_0^{2\pi}\frac{2f(R)}{R}\mathcal{W}_{I_1}(I_1(R))\delta f(R,\Theta)d\Theta dR\,. \tag{4.18}$$

[8]Note that since $\delta z=\delta z(R,\Theta)$ is periodic with repect to Θ, one has $\int_0^{2\pi}\delta z_{,\Theta}(R,\Theta)d\Theta=\delta z(R,2\pi)-\delta z(R,0)=0$.

Remark 4.1. *Note that* (4.18) *can be written as*

$$\delta W = \int_0^{R_o} \frac{4\pi f(R)}{R} \mathcal{W}_{I_1}(I_1(R))\delta f_0(R)dR\,, \tag{4.19}$$

where $\delta f_0(R) = \frac{1}{2\pi}\int_0^{2\pi} \delta f(R,\Theta)d\Theta$ *is the angular mean value of* δf *. On the other hand, one can write*

$$\delta f_0(R) = \frac{1}{2\pi}\int_0^{2\pi} \delta f(R,\Theta)d\Theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{R} \xi\delta b(\xi,\Theta)d\xi d\Theta = \frac{1}{2\pi}\int_0^{R}\xi\delta b_0(\xi)d\xi,$$

where

$$\delta b_0(R) = \frac{1}{2\pi}\int_0^{2\pi}\delta b(R,\Theta)d\Theta. \tag{4.20}$$

Hence, the energy variation depends only on $\delta b_0(R)$ *—the angular mean value of the perturbation* $\delta b(R,\Theta)$.

(d) Perturbed dislocations in incompressible neo-Hookean solids

Let us consider an incompressible homogeneous neo-Hookean solid, i.e., $\bar{\mathcal{W}}(I_1) = \frac{\mu}{2}(I_1 - 3)$, where μ is the shear modulus for infinitesimal strains, and an arbitrary perturbation $\delta b = \delta b(R,\Theta)$.

Remark 4.2. *Note that even though the energy per unit length along a single screw dislocation line in a neo-Hookean solid is unbounded as shown in [25] (see also [44]), energy is not necessarily unbounded for distributed screw dislocations. In particular, for a radially-symmetric distribution of screw dislocations, the energy per unit length in a neo-Hookean solid is written as*

$$W = 2\pi\int_0^{R_o}\frac{\mu}{2}(I_1(\xi)-3)\,\xi d\xi = \pi\mu\int_0^{R_o}\frac{f^2(\xi)}{\xi}d\xi\,.$$

Let us assume, as an example for computing the energy, the following Burgers' vector distribution

$$b(R) = \begin{cases} b_i & 0 < R \leq R_i\,,\\ 0 & R_i < R \leq R_o\,,\end{cases} \tag{4.21}$$

where $R_i \leq R_o$ *is the radius of a cylinder made of a solid with a uniform Burgers' vector* b_i *, while the hollow cylinder* $R_i < R \leq R_o$ *is dislocation-free. Thus, one finds*

$$f(\xi) = \frac{1}{2\pi}\int_0^{\xi}\zeta b(\zeta)d\zeta = \begin{cases}\dfrac{b_i\xi^2}{4\pi} & 0 \leq \xi \leq R_i\,,\\[2mm] \dfrac{b_iR_i^2}{4\pi} & R_i < \xi \leq R_o\,.\end{cases} \tag{4.22}$$

Therefore

$$W = \pi\mu\int_0^{R_i}\frac{1}{\xi}\left(\frac{b_i\xi^2}{4\pi}\right)^2 d\xi + \pi\mu\int_{R_i}^{R_o}\frac{1}{\xi}\left(\frac{b_iR_i^2}{4\pi}\right)^2 d\xi = \frac{\mu b_i^2R_i^4}{64\pi}\left[1+4\log\left(\frac{R_o}{R_i}\right)\right] < \infty\,.$$

In the following computation, we consider an arbitrary radially-symmetric Burgers' vector distribution $b = b(R)$ and an arbitrary perturbation $\delta b = \delta b(R,\Theta)$. For a neo-Hookean solid, the ordinary differential equations (4.16) for $k\in\mathbb{Z}$ simplify and read

$$R^2\delta z_k{}'' + R\delta z_k{}' - k^2\delta z_k = \mathrm{i}k\delta f_k\,. \tag{4.23}$$

Solving (4.23), one finds that for $k\in\mathbb{Z}$

$$\begin{aligned}\delta z_k(R) = \frac{R^{2k}+R_o^{2k}}{2R^kR_o^k}&\left[c_k + \mathrm{i}\int_{\frac{R}{R_o}}^{1}\frac{\left(\xi^k-\xi^{-k}\right)\delta f_k(R_o\xi)}{2\xi}d\xi\right]\\ &+\frac{R^{2k}-R_o^{2k}}{2R^kR_o^k}\left[d_k - \mathrm{i}\int_{\frac{R}{R_o}}^{1}\frac{\left(\xi^k+\xi^{-k}\right)\delta f_k(R_o\xi)}{2\xi}d\xi\right],\end{aligned} \tag{4.24}$$

for some complex constants c_k and d_k. By using the boundary condition (4.17) $\delta z_k{}'(R_o)=0$, it follows that $d_k=0$. We observe that $c_k=\delta z_k(R_o)$, and from (4.15), one observes that $\delta z_{-k}=\delta z_k^*$.[9] Thus, $c_{-k}=c_k^*$. Also, note that $\delta f_{-k}=\delta f_k^*$. Therefore, following (4.24) and by using (4.14), it follows that

$$\begin{aligned}\delta z(R,\Theta) = c_0 + \sum_{k=1}^{\infty}\frac{R^{2k}+R_o^{2k}}{2R^kR_o^k}&\Bigg[2\Big(\Re(c_k)\cos(k\Theta)-\Im(c_k)\sin(k\Theta)\Big)\\ &-\int_{\frac{R}{R_o}}^{1}\frac{\left(\xi^k-\xi^{-k}\right)\Big(\Re[\delta f_k(R_o\xi)]\sin(k\Theta)+\Im[\delta f_k(R_o\xi)]\cos(k\Theta)\Big)}{\xi}d\xi\Bigg]\\ &+\sum_{k=1}^{\infty}\frac{R^{2k}-R_o^{2k}}{2R^kR_o^k}\int_{\frac{R}{R_o}}^{1}\frac{\left(\xi^k+\xi^{-k}\right)\Big(\Re[\delta f_k(R_o\xi)]\sin(k\Theta)+\Im[\delta f_k(R_o\xi)]\cos(k\Theta)\Big)}{\xi}d\xi\,,\end{aligned} \tag{4.25}$$

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of a complex number z, respectively. Note that since $\delta f_k(R)=\frac{1}{2\pi}\int_0^{2\pi}\delta f(R,\zeta)e^{-\mathrm{i}k\zeta}d\zeta$, one can write

$$\Re\left[\delta f_k(R)\right] = \frac{1}{2\pi}\int_0^{2\pi}\delta f(R,\Theta)\cos(k\Theta)\,d\Theta\,,\quad \Im\left[\delta f_k(R)\right] = -\frac{1}{2\pi}\int_0^{2\pi}\delta f(R,\Theta)\sin(k\Theta)\,d\Theta\,.$$

Remark 4.3. *Note that for a neo-Hookean solid, the incremental deformation is independent of the finite radially-symmetric dislocation distribution $b=b(R)$. Indeed, the governing equation (4.23) holds for any $b=b(R)$. However, as can be seen in (4.13), the incremental stress field, and in particular $\delta\sigma^{zz}$, depends on the initial dislocation distribution.*

Let us now simplify the solution (4.25) for a particular Burgers' vector perturbation given by

$$\delta b(R,\Theta) = \delta b_0(R) + \frac{R}{R_o}\left(1-\frac{R}{R_o}\right)^2\left[b_1\cos\Theta + b_2\sin\Theta\right]\,, \tag{4.26}$$

for some R-dependent function $\delta b_0=\delta b_0(R)$, and constants b_1 and b_2. Note that the only non-zero Fourier coefficients of δb in (4.26) are δb_0, δb_1, and δb_{-1}. For $k=-1,1$, one finds

$$\delta b_k(R) = \frac{1}{2}\left(b_1-\mathrm{i}kb_2\right)\frac{R}{R_o}\left(1-\frac{R}{R_o}\right)^2\,.$$

Therefore, the only non-zero Fourier coefficients of δf are δf_0, δf_1, and δf_{-1}. They read

$$\delta f_0(R) = \frac{1}{2\pi}\int_0^R\xi\delta b_0(\xi)d\xi \quad\text{and}\quad \delta f_k(R) = \frac{b_1-\mathrm{i}kb_2}{4\pi}\left(\frac{R^5}{5R_o^3}-\frac{R^4}{2R_o^2}+\frac{R^3}{3R_o}\right)\quad\text{for}\quad k=-1,1\,. \tag{4.27}$$

First, note that following (4.24), for $k\neq-1,1$ one obtains $\delta z_k(R)=c_k\frac{R^{2k}+R_o^{2k}}{2R^kR_o^k}$. However, since we are looking for a solution that is bounded, it follows that for $k\neq-1,0,1$, one has $c_k=0$.

[9] We denote by x^* the complex conjugate of a complex number x.

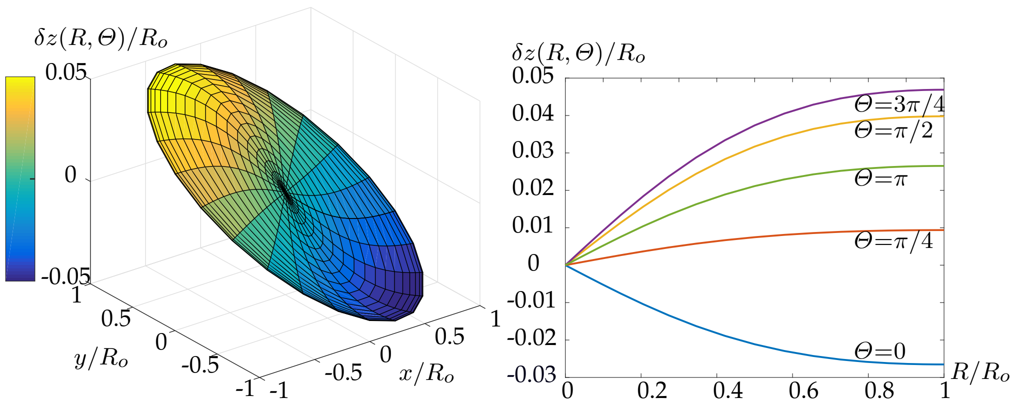

Figure 1. Visualization of the solution (4.29) for a cylinder of radius R_o with $b_1 R_o = 15$ and $b_2 R_o = 10$. Left: 3D visualization of the deformation of a cross section of the cylinder. Right: Profile of deformation of different radial lines.

Thus, one finds following (4.25) that

$$\begin{aligned}\delta z(R,\Theta) = c_0 &- \frac{b_1\,(R-R_o)^2\left(R^4-2R_oR^3+2R_o^3R+R_o^4\right)+240\pi\Im(c_1)R_o^2\left(R^2+R_o^2\right)}{240\pi R_o^3 R}\sin\Theta\\ &+\frac{b_2\,(R-R_o)^2\left(R^4-2R_oR^3+2R_o^3R+R_o^4\right)+240\pi\Re(c_1)R_o^2\left(R^2+R_o^2\right)}{240\pi R_o^3 R}\cos\Theta\,.\end{aligned} \tag{4.28}$$

Further, to ensure that (4.28) is bounded, one must have $b_1 R_o^2 + 240\pi\Im(c_1) = 0$, and $b_2 R_o^2 + 240\pi\Re(c_1) = 0$. Thus, $\Im(c_1) = -b_1 R_o^2/(240\pi)$, and $\Re(c_1) = -b_2 R_o^2/(240\pi)$. Next, by enforcing the boundary condition (4.17) $\delta z(0,\Theta) = 0$ to fix the rigid body motion, one finds $c_0 = 0$. Therefore, it follows that

$$\delta z(R,\Theta) = \frac{b_2\cos\Theta - b_1\sin\Theta}{240\pi R_o^3} R\left(R^4 - 4R^3R_o + 5R^2R_o^2 - 4R_o^4\right). \tag{4.29}$$

In Fig. 1, we plot the solution (4.29) for a cylinder of radius R_o subject to a perturbation (4.26) such that $b_1 R_o = 15$ and $b_2 R_o = 10$. Note that the numerical values shown in Fig. 1 should be multiplied by a small ϵ to give the incremental deformation. Given that $z = Z$ for the finite dislocation distribution, the total deformation reads: $z_\epsilon = Z + \epsilon\delta z + o(\epsilon)$. Recall, as noted earlier, that the state of deformation of a cylinder made of a neo-Hookean solid is independent of $b = b(R)$; it only depends on the perturbation —compare this to Example (e) where the deformation of a cylinder made of a power law material actually depends on the finite dislocation distribution $b = b(R)$.

Using (4.1) and (4.13), one finds the following total stress in the perturbed configuration (recall that the total stress in the perturbed configuration for a small enough ϵ is $\boldsymbol{\sigma}_\epsilon = \boldsymbol{\sigma} + \epsilon\delta\boldsymbol{\sigma} + o(\epsilon)$.)

$$\boldsymbol{\sigma}_\epsilon = \begin{pmatrix} 0 & 0 & \epsilon\mu\delta z_{,R} \\ 0 & 0 & -\mu\frac{f(R)}{R^2} - \epsilon\frac{\mu}{R^2}\left(\delta f - \delta z_{,\Theta}\right) \\ \epsilon\mu\delta z_{,R} & -\mu\frac{f(R)}{R^2} - \epsilon\frac{\mu}{R^2}\left(\delta f - \delta z_{,\Theta}\right) & \mu\frac{f^2(R)}{R^2} + \epsilon\frac{2\mu}{R^2} f\delta f \end{pmatrix} + o(\epsilon)\,,$$

where

$$\delta f = \frac{1}{2\pi}\int_0^R \xi\delta b_0(\xi)d\xi + \left(\frac{R^5}{5R_o^3} - \frac{R^4}{2R_o^2} + \frac{R^3}{3R_o}\right)\frac{b_1\cos\Theta + b_2\sin\Theta}{2\pi}\,,$$

$$\delta z_{,R} = \left(5R^4 - 16R^3R_o + 15R^2R_o^2 - 4R_o^4\right)\frac{b_1\sin\Theta - b_2\cos\Theta}{240\pi R_o^3}\,,$$

$$\delta f - \delta z_{,\Theta} = \frac{1}{2\pi}\int_0^R \xi\delta b_0(\xi)d\xi + R\left(23R^4 - 56R^3R_o + 35R^2R_o^2 + 4R_o^4\right)\frac{b_1\cos\Theta + b_2\sin\Theta}{240\pi R_o^3}\,.$$

Let us now compute the variation of the energy due to a dislocation distribution perturbation. Following (4.19), one has

$$\delta W = \int_0^{R_o}\frac{2\pi\mu}{R}f(R)\delta f_0(R)dR\,.$$

Assuming the finite dislocation distribution (4.21), the variation of the energy reads

$$\delta W = \int_0^{R_i}\frac{\mu b_i R}{4\pi}\int_0^R \xi\delta b_0(\xi)d\xi dR + \int_{R_i}^{R_o}\frac{\mu b_i R_i^2}{4R\pi}\int_0^R \xi\delta b_0(\xi)d\xi dR\,.$$

Let us assume that the total Burgers' vector of the perturbation is zero so that the perturbation does not change the total Burgers' vector of the original finite dislocation distribution $b(R)$, i.e., $\int_0^{R_o}\int_0^{2\pi} R\delta b(R,\Theta)d\Theta dR = 0$. In terms of the angular mean value of the perturbation this is written as $\int_0^{R_o} R\delta b_0(R)dR = 0$. We consider in particular a Burgers' vector perturbation such that its angular mean value —cf. (4.20) —is given by

$$\delta b_0(R) = 15 b_0\frac{R}{R_o}\left(1 - \frac{R}{R_o}\right)^2\left(1 - 2\frac{R}{R_o}\right)\,, \tag{4.30}$$

for some constant b_0. For this perturbation one obtains

$$\delta W = \frac{\mu b_i b_o\left(35R_i^8 - 144R_i^7R_o + 210R_i^6R_o^2 - 112R_i^5R_o^3 + 14R_i^2R_o^6\right)}{672\pi R_o^4}\,.$$

Note that for any R_i such that $0 < R_i < R_o$, the energy variation δW has the same sign as $b_i b_o$. For $R_i = 0$, $\delta W = 0$ and $\delta W/(b_i b_o)$ is monotonically increasing as a function of R_i. In particular, for $R_i > 0$, $\delta W \neq 0$, and hence the initial dislocation distribution is not in equilibrium.

(e) Perturbed dislocations in incompressible power law solids

Let us consider an arbitrary perturbation $\delta b = \delta b(R,\Theta)$ in the case of an incompressible power law solid for which the energy function is written as

$$\bar{\mathcal{W}}(I_1) = \frac{\mu}{2c}\left\{\left[1 + \frac{c}{n}(I_1 - 3)\right]^n - 1\right\}\,, \tag{4.31}$$

where μ is the shear modulus for infinitesimal strains, n is a hardening exponent, and c is another material constant. Based on the work of Knowles [45] on anti-plane shear fields, Rosakis and Rosakis [26] observed that when $n = \frac{1}{2}$, the energy per unit length along a single screw dislocation line is finite. We assume in what follows that $n = \frac{1}{2}$ and $c = 1$. For such a power law material, the ordinary differential equation (4.16) for $k \in \mathbb{Z}$ is simplified to read

$$\left(2Rf^2 + R^3\right)\delta z_k'' + \left(R^2 + 4f^2 - 2Rff'\right)\delta z_k' - k^2R\delta z_k = ikR\delta f_k\,, \tag{4.32}$$

along with the boundary conditions (4.17): $\delta z_k(0) = 0$, $\delta z_k'(R_o) = 0$. In this example we assume the Burgers' vector distribution (4.21) and the Burgers' vector perturbation (4.26). Therefore, f and the non-zero Fourier coefficients of δf are again given by (4.22) and (4.27), respectively. We numerically solve (4.32) and in Fig. 2 plot the profile of the deformation $\delta z = \delta z(R,\Theta)$ of a cross section of a cylinder of radius R_o with the finite dislocation distribution (4.21) such that $b_i R_o = 25$

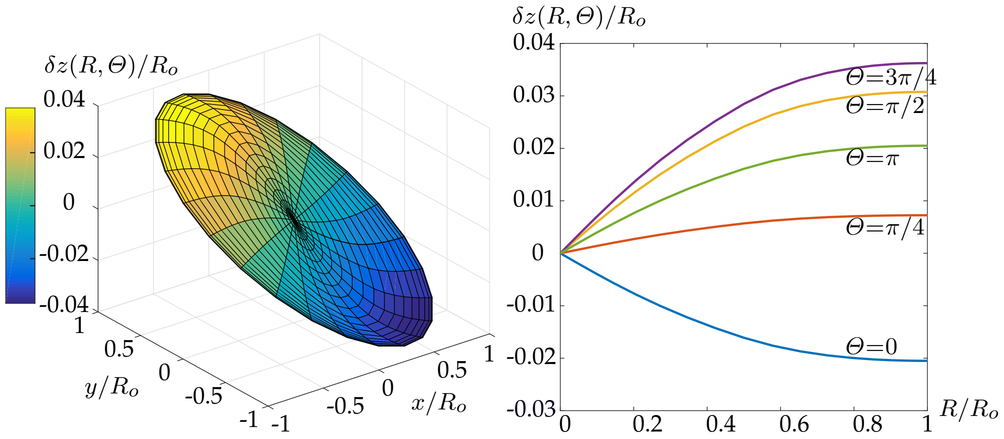

Figure 2. Visualization of the deformation $\delta z = \delta z(R,\Theta)$—solution of (4.32)—of a cylinder of radius R_o with the finite dislocation distribution (4.21) such that $b_i R_o = 25$ and $R_i/R_o = 0.5$, and subject to the Burgers' vector perturbation (4.26) such that $b_1 R_o = 15$ and $b_2 R_o = 10$. Left: 3D visualization of the deformation of a cross section of the cylinder. Right: Profile of deformation of different radial lines.

and $R_i/R_o = 0.5$, and subject to the Burgers' vector perturbation (4.26) such that $b_1 R_o = 15$ and $b_2 R_o = 10$.

The total stress in the perturbed configuration is computed following (4.1), and (4.13). Its non-zero components read (Recall that f is given by (4.22).)

$$\begin{aligned}
\sigma_\epsilon^{rz} &= \frac{\epsilon\mu}{\sqrt{\frac{2f(R)^2}{R^2}+1}}\delta z_{,R} + o(\epsilon)\,,\\
\sigma_\epsilon^{\theta z} &= -\frac{\mu f(R)}{R^2\sqrt{\frac{2f(R)^2}{R^2}+1}} - \epsilon\mu\left[\frac{1}{R^2\sqrt{\frac{2f(R)^2}{R^2}+1}} - \frac{2f^2}{R^4\left(\frac{2f(R)^2}{R^2}+1\right)^{3/2}}\right]\left(\delta f - \delta z_{,\Theta}\right) + o(\epsilon)\,,\\
\sigma_\epsilon^{zz} &= \frac{\mu f^2(R)}{R^2\sqrt{\frac{2f(R)^2}{R^2}+1}} + \epsilon\frac{2\mu f}{R^2\sqrt{\frac{2f(R)^2}{R^2}+1}}\delta f - \epsilon\frac{2\mu f^3}{R^4\left(\frac{2f(R)^2}{R^2}+1\right)^{3/2}}\left(\delta f - \delta z_{,\Theta}\right) + o(\epsilon)\,.
\end{aligned} \tag{4.33}$$

The variation of the energy is written as

$$\delta W = \int_0^{R_i} \frac{\mu b_i R}{4\pi\sqrt{\frac{b_i^2 R^2}{8\pi^2}+1}} \int_0^R \xi\delta b_0(\xi)d\xi dR + \int_{R_i}^{R_o} \frac{\mu b_i R_i^2}{4R\pi\sqrt{\frac{b_i^2 R_i^4}{8\pi^2 R^2}+1}} \int_0^R \xi\delta b_0(\xi)d\xi dR\,.$$

Assuming a dislocation perturbation with a vanishing total Burgers' vector such that its mean angular value is given by (4.30) one can compute the energy variation and find that it is not zero, i.e., the initial dislocation distribution is not in equilibrium.

5. Conclusion

In this paper we introduce a geometric theory of small-on-large anelasticity to study the induced small deformations due to a perturbation of a given distribution of (finite) eigenstrains superposed on the finite deformation that corresponds to the original distribution. Given a nonlinear solid with a given distribution of eigenstrains, a perturbation of the eigenstrains changes the equilibrium configuration and its state of stress. In the geometric formulation of anelasticity, a perturbation of the anelasticity source corresponds to a perturbation of the geometry of the material manifold. We find the incremental residual stresses due to the perturbation fields and derive the governing equations for the induced small deformations superposed on the original finite deformation. Finally, to illustrate the capability of the theory, we consider an axi-symmetric distribution of parallel screw dislocations in an incompressible isotropic solid and calculate the perturbation fields when the body undergoes an arbitrary small perturbation in the Burgers' vector distribution. For generalized neo-Hookean solids, we are able to reduce the governing equations to a single ordinary differential equation. Furthermore, when the solid is neo-Hookean, we find a closed-form solution for the governing equations. We also consider the power law solid constitutive model for which we solve the governing equations numerically.

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