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On the origins of the idea of the multiplicative decomposition of the deformation gradient

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The objective of this short note is to trace back the major contributions that led to the multiplicative decomposition of the deformation gradient in finite plasticity, nonlinear thermoelasticity, and growth mechanics. In the 1940s, Eckart in the US and Kondo in Japan independently paved the road to the formulation of a nonlinear theory capable of modeling anelastic phenomena. As opposed to assuming, for a given body, the existence of a global stress-free configuration (the "principle of relaxability-in-the-large" according to Eckart) that the body takes whenever it is completely relaxed, Eckart [1] suggested an alternative framework for anelasticity based on what he called "relaxability-in-the-small". He conceptually constructed a local stress-free "fragmented" state following a local relaxation of the reference configuration by "cutting out" a "small bit of matter" around every material point and letting it relax independently of the remainder of the body. He also asserted that such a construction should be accompanied by an elastic deformation to ensure that the body keeps its structural integrity. This is nothing but the decomposition of the deformation gradient into an anelastic relaxation, leading to the so-called "intermediate" configuration, followed by the elastic portion of the deformation gradient.

Independently of Eckart's work, Kondo [2] observed that due to plastic deformations, the relaxed state of a body has a non-trivial geometry that is not compatible with that of the Euclidean space. This observation first led him to construct a stress-free configuration as a Riemannian manifold in which a non-vanishing curvature is a measure of the incompatibility of the plastic deformation. Inspired by the works of Cartan [3, 4] on non-trivial holonomy groups, Kondo [5–7] extended his framework to consider the material body as a non-Riemannian space with a non-zero torsion. He used this geometric framework in the context of crystals with geometrical imperfections, e.g. dislocations, and introduced the idea of considering the stress-free state as "an amorphous aggregation" of small pieces of relaxed perfect "crystalline pieces" that he modeled as a non-Riemannian manifold. Further, he interpreted the torsion tensor as a measure of the density of dislocations and initiated the development of a geometric theory of dislocation mechanics. Soon after, further contributions to the nonlinear theory of dislocation mechanics were introduced by Kröner [8, 9] and Bilby et al. [10]. For a review of the interactions between the Japanese (led by Kondo), the British (led by Bilby), and the German (led by Kröner) schools and their contributions, see [11]. It is worth mentioning that Sedov [12] independently realized that a body in plastic deformation can be relaxed in a stress-free intermediate configuration, which he called "a new starting position", with a changing metric that is generally non-Euclidean.

Following the original idea of local relaxation inspired by the pioneering works cited above, the first formal introduction of the multiplicative decomposition of the deformation gradient in finite plasticity appeared in the

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late 1950s in the work of Bilby et al. [13]. Bilby et al. [13] called the total deformation gradient $\mathbf{F}$, the elastic deformation gradient $\mathbf{F}_e$, and the plastic deformation gradient $\mathbf{F}_p$, “shape deformation, “lattice deformation”, and “dislocation deformation”, respectively. The decomposition $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ was explicitly written in [13, p. 41, Equation (12)]. The same decomposition is seen in the work of Kröner [14, p. 286, Equation (4)] as well. Almost a decade later, Lee and Liu [15, 16] discussed the multiplicative decomposition in finite plasticity and received most of the credit for it. In nonlinear thermoelasticity, the first formal introduction of the multiplicative decomposition of the deformation gradient is due to Stojanović et al. [17, 18]. In the biomechanics and growth mechanics literature, the introduction of the multiplicative decomposition is usually attributed to Rodriguez et al. [19]. However, it was first introduced about a decade earlier independently in Russia by Kondaurov and Nikitin [20] and in Japan by Takamizawa et al. [21–23].

References

[1] Eckart, C. The thermodynamics of irreversible processes. IV. The theory of elasticity and anelasticity. *Phys Rev* 1948; 73(4): 373.

[2] Kondo, K. A proposal of a new theory concerning the yielding of materials based on Riemannian geometry, I. *J Soc Appl Mech Japan* 1949; 2(11): 123–128.

[3] Cartan, É. Les groupes d’holonomie des espaces généralisés. *Acta Mathematica* 1926; 48(1–2): 1–42.

[4] Cartan, É. *Leçons sur la géométrie des espaces de Riemann*. Paris: Gauthier-Villars, 1928.

[5] Kondo, K. On the dislocation, the group of holonoray and the theory of yielding. *J Soc Appl Mech Japan* 1950; 3(17): 107–110.

[6] Kondo, K. On the fundamental equations of the theory of yielding. *J Soc Appl Mech Japan* 1950; 3(20): 184–188.

[7] Kondo, K. On the geometrical and physical foundations of the theory of yielding. In *Proceedings of the 2nd Japan national congress on applied mechanics*, volume 2, pp. 41–47.

[8] Kröner, E. Die inneren spannungen und der inkompatibilitatstensor in der elastizitatstheorie. *Z angew Phys* 1955; 7(5): 249–257.

[9] Kröner, E, and Seeger, A. Nicht-lineare elastizitätstheorie der versetzungen und eigenspannungen. *Arch Rat Mech Anal* 1959; 3(1): 97–119.

[10] Bilby, B, Bullough, R, and Smith, E. Continuous distributions of dislocations: a new application of the methods of non-riemannian geometry. *Proc R Soc Lond A* 231: 263–273.

[11] Kondo, K. On the analytical and physical foundations of the theory of dislocations and yielding by the differential geometry of continua. *Int J Eng Sci* 1964; 2(3): 219–251.

[12] Sedov, LI. *Introduction to the Mechanics of a Continuous Medium* [First published in Russian in 1962]. Reading, MA: Addison-Wesley, 1965.

[13] Bilby, BA, Lardner, LRT, and Stroh, AN. Continuous distributions of dislocations and the theory of plasticity. In *Actes du IXe congrès international de mécanique appliquée*, Bruxelles, 1956, volume 8, pp. 35–44.

[14] Kröner, E. Allgemeine kontinuumstheorie der versetzungen und eigenspannungen. *Arch Rat Mech Anal* 1959; 4(1): 273–334.

[15] Lee, E and Liu, D. Finite-strain elastic–plastic theory with application to plane-wave analysis. *J Appl Phys* 1967; 38(1): 19–27.

[16] Lee, EH. Elastic–plastic deformation at finite strains. *J Appl Mech* 1969; 36(1): 1–6.

[17] Stojanović, R, Djurić, S, and Vujošević, L. On finite thermal deformations. *Archiwum Mechaniki Stosowanej* 1964; 1(16): 103–108.

[18] Stojanović, R. On the stress relation in non-linear thermoelasticity. *Int J Non-Lin Mech* 1969; 4(3): 217–233.

[19] Rodriguez, EK, Hoger, A, and McCulloch, AD. Stress-dependent finite growth in soft elastic tissues. *J Biomech* 1994; 27(4): 455–467.

[20] Kondaurov, V, and Nikitin, L. Finite strains of viscoelastic muscle tissue. *J Appl Math Mech* 1987; 51(3): 346–353.

[21] Takamizawa, K, and Hayashi, K. Strain energy density function and uniform strain hypothesis for arterial mechanics. *J Biomech* 1987; 20(1): 7–17.

[22] Takamizawa, K, and Matsuda, T. Kinematics for bodies undergoing residual stress and its applications to the left ventricle. *J Appl Mech* 1990; 57(2): 321–329.

[23] Takamizawa, K. Stress-free configuration of a thick-walled cylindrical model of the artery: an application of riemann geometry to the biomechanics of soft tissues. *J Appl Mech* 1991; 58(3): 840–842.