Nonlinear Elasticity in a Deforming Ambient Space

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Abstract

In this paper we formulate a nonlinear elasticity theory in which the ambient space is evolving. For a continuum moving in an evolving ambient space, we model time dependency of the metric by a time-dependent embedding of the ambient space in a larger manifold with a fixed background metric. We derive both the tangential and the normal governing equations. We then reduce the standard energy balance written in the larger ambient space to that in the ambient space with an evolving metric. We consider quasi-static deformations of the ambient space and show that a quasi-static deformation of ambient space results in stresses, in general. We linearize the nonlinear theory about a reference motion and show that variation of the spatial metric corresponds to an effective field of body forces.

Keywords: Geometric mechanics, nonlinear elasticity, deforming ambient space.

Contents

1 Introduction

In the geometric theory of elasticity, an elastic body is represented by a *material manifold*, $\mathcal{B}$, which defines the natural, stress-free state of the body. The body moves in an ambient space, which in turn is represented by a *spatial manifold*, $\mathcal{S}$. The motion of the body is described by a time-dependent configuration map $\varphi_t : \mathcal{B} \to \mathcal{S}$ from the material manifold to the spatial manifold.

The manifolds $\mathcal{B}$ and $\mathcal{S}$ are not simply differential manifolds, but have further geometric structures that allow one to measure the amount of stretch in the body for a given configuration. While more general geometric structures involving torsion and non-metricity are used in general (see [Yavari and Goriely, 2012a,b,c, 2014] for

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recent work), in this paper, we restrict our attention to Riemannian manifolds: we assume that both $\mathcal{B}$ and $\mathcal{S}$ are Riemannian, with metric tensors $\boldsymbol{G}$ and $\boldsymbol{g}$, and the associated Levi-Civita connections $\nabla^{\boldsymbol{G}}$ and $\nabla^{\boldsymbol{g}}$, respectively. For a given configuration, these Riemannian structures allow one to evaluate the spatial distances between the points of the body, and distances given by the material metric. A discrepancy between these two types of distances signifies a strain in the body from its natural, stress-free state, and hence results in stresses. The material and spatial distances can be unequal even for a body at rest, without any external forces applied—this is the case of residual stresses. This viewpoint has been explored for thermal stresses and for growing bodies in [Ozakin and Yavari, 2010; Sadik and Yavari, 2015] and [Yavari, 2010; Sadik et al., 2015], respectively. Certain non-elastic deformations of a material can be described by a time-dependent material metric, $\boldsymbol{g}_t$; for the case of thermal expansion, the material metric has been related to the (possibly evolving) temperature distribution in [Ozakin and Yavari, 2010; Sadik and Yavari, 2015].

In this paper, we take the material metric as fixed, but consider an evolving spatial metric, $\boldsymbol{g}_t$. In the geometric field theory of elasticity [Marsden and Hughes, 1983; Simo and Marsden, 1984b; Yavari et al., 2006], the spatial metric $\boldsymbol{g}_t$ is introduced as a fixed background geometry. Likewise, in the classical theory of nonlinear elasticity, this background metric is a given geometric object with no dynamics; $\boldsymbol{g}_t$ is "absolute" in the sense of Anderson [1967] and is a "structural field" in the sense of Post [1997]. Our motivation to study the effects of a time-dependent spatial geometry stems both from the hope of a deeper understanding of the structure of the classical theory,[1] and from possible applications involving the analysis of elastic bodies constrained to move on curved, dynamical surfaces. In order to have a sense of what to expect from such a theory, let us consider a simple, two-dimensional example.

Fig. 1 shows a thin elastic strip constrained to move on the surface of a torus, which we treat as the ambient space, $\mathcal{S}$. We assume that there is no friction between the torus and the strip—the latter moves freely on the torus, but cannot move away from it.[2] Suppose that the torus is expanding in time in a pre-determined manner, and consider the motion of the strip in this dynamic ambient space. Our aim is to investigate this dynamics in terms of the intrinsic geometry of $\mathcal{S}$ and forces that "live" on this surface, to the extent possible. It is evident that the strip will stretch and thus will store elastic potential energy as the torus expands, even though it does not see any sources of external forces in the ambient space it observes. Thus, the energy balance written inside the torus (without any reference to the surrounding Euclidean $\mathbb{R}^3$) will suggest a non-conservation of energy. In this example, we know the missing piece in the energy balance; it is simply the work done by the normal forces needed to expand the torus with the strip on it. Our aim is to obtain general equations describing the motion of an elastic body moving in such an evolving ambient space, and investigate such issues as energy balance and Lagrangian mechanics.

Implicit in this discussion of Fig. 1 is the fact that the ambient space $\mathcal{S}$ is considered as an embedded submanifold of $\mathbb{R}^3$, instead of simply as a Riemannian manifold in its own right. While it is possible to consider the dynamics of a body in an ambient space which is described purely intrinsically in terms of the time-dependent metric $\boldsymbol{g}_t$, we will see that there are several advantages to considering the motion in terms of a time-dependent embedding in a larger, static space. For one, it will be possible to identify the missing part in the intrinsic energy balance as work done by/on the outside forces trasforming the ambient space $\mathcal{S}$, as the discussion above suggests. In addition, the dynamics in a time-dependent submanifold also results in fictitious forces that cannot be obtained purely from the intrinsic geometry—forces that depend explicitly on the embedding. We will identify the effects of the intrinsic and the extrinsic geometries of $\mathcal{S}$ below.

The notion of a time-dependent ambient space also shows up in the theory of general relativity. While there are connections between the theory of elasticity developed in this paper and general relativistic continuum mechanics, we will leave the discussion of these issues to a future communication, focusing on the non-relativistic case exemplified by the case of Fig. 1 in this paper.

This paper is organized as follows. In §2, we first formulate a Lagrangian field theory of elasticity when ambient space has a time-dependent background metric. We then show that for an elastic body moving in an evolving ambient space energy balance must be modified and obtain a modified energy balance when spatial metric is time dependent. We do this by considering a time-dependent embedding of the ambient space in a

[1] The generalization of a theory obtained by relaxing certain standard assumptions (in this case, the staticity of $\boldsymbol{g}_t$), commonly results in a deeper understanding of the original theory. Examples of this include the geometric notions of stress and traction obtained by allowing the spatial metric to be non-euclidean.

[2] It may be helpful to imagine the strip as moving between two tori of infinitesimally different sizes, so that the strip is constrained from both sides.

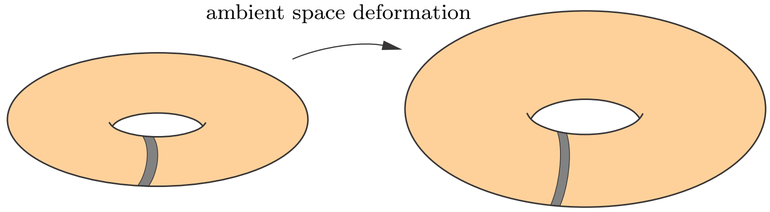

Figure 1.1: *A thin strip embedded in a toroidal ambient space is stretched when the ambient space deforms.*

larger manifold and the standard energy balance written in the larger manifold. We reduce the energy balance to that written by an observer in the evolving ambient space. We then study consequences of covariance of energy balance when ambient space deforms in time. In §3 we look at quasi-static deformations of the ambient space and the induced stresses. We linearize the governing equations and show that a quasi-static deformation of the ambient space is equivalent to an effective body force in a fixed ambient space.

2 Motion of an Elastic Body in an Evolving Ambient Space

In this section, we study the motion of an elastic body moving in an evolving (time-dependent) ambient space. We will derive both tangential and normal governing equations of motion, balance of mass, and energy balance.

2.1 Lagrangian Field Theory of Elasticity in an Evolving Ambient Space

We identify the reference configuration of an elastic body with a Riemannian manifold $(\mathcal{B},\boldsymbol{G})$ and let the body deform in a time-dependent ambient space $\mathcal{S}_t$, which is evolving in a Euclidean space $(\mathcal{Q},\boldsymbol{h})$ of higher dimension. The evolution of the ambient space $\mathcal{S}_t$ is given by a time-dependent embedding $\psi_t:\mathcal{S}\to\mathcal{Q}$, for some manifold $\mathcal{S}$,[3] such that $\psi_t(\mathcal{S})=\mathcal{S}_t$, and the evolving metric of the ambient space $\mathcal{S}$ is given as the induced metric by that of $\mathcal{Q}$, i.e., $\boldsymbol{g}_t:=\psi_t^*\boldsymbol{h}$, i.e., ψ_t is an isometric embedding.[4] See Fig. 2.1. We denote inner products of vectors with respect to the metrics $\boldsymbol{h}$ and $\boldsymbol{g}_t$ by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{h}}$ and $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{g}_t}$, respectively. Let $\dim\mathcal{S}_t=n$, and $\dim\mathcal{Q}=n+k=m$. Let $\{\boldsymbol{\eta}_i\}_{i=1,\ldots,k}$ be a smooth orthonormal basis for $\mathfrak{X}^{\perp}(\mathcal{S}_t)$, the set of vector fields normal to $\mathcal{S}_t$. Recall as discussed in Appendix A that every vector field $\boldsymbol{u}$ on $\mathcal{S}_t$ can be written as $\boldsymbol{u}=\boldsymbol{u}_{\parallel}+\sum_{i=1}^{k}u^i\boldsymbol{\eta}_i$, where $(.)_{\parallel}$ is the tangential part of the vector $(.)$ with respect to $\mathcal{S}_t$, and that for $i,j\in\{1,\ldots,k\}$, we have $\langle\!\langle\boldsymbol{\eta}_i,\boldsymbol{\eta}_j\rangle\!\rangle_{\boldsymbol{h}}=\delta_{ij}$ and $\langle\!\langle\boldsymbol{\eta}_i,\boldsymbol{u}_{\parallel}\rangle\!\rangle_{\boldsymbol{h}}=0$. For $i\in\{1,\ldots,k\}$, we denote the i^{th} second fundamental form of $\mathcal{S}_t$ along the i^{th} unit normal $\boldsymbol{\eta}_i$ by $\boldsymbol{k}_{(i)t}$. We define a motion of $(\mathcal{B},\boldsymbol{G})$ in $\left(\mathcal{S}_t,\boldsymbol{h}|_{\mathcal{S}_t}\right)$ as a one-parameter family of maps $\tilde{\varphi}_t:\mathcal{B}\to\mathcal{S}_t$, where t is time, which is equivalent to a motion of $(\mathcal{B},\boldsymbol{G})$ in $(\mathcal{S},\boldsymbol{g}_t)$: $\varphi_t:\mathcal{B}\to\mathcal{S}$, such that $\varphi_t=\psi_t^{-1}\circ\tilde{\varphi}_t$. We let $\tilde{\varphi}(X,t):=\tilde{\varphi}_t(X)$, $\varphi(X,t):=\varphi_t(X)$, and $\psi(x,t):=\psi_t(x)$. We denote the local coordinates on $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{Q}$ by $\{X^A\}$, $\{x^a\}$, and $\{\chi^\alpha\}$, respectively. Let $\{\chi^\alpha\}$ be a local coordinate chart for $\mathcal{Q}$ such that at any point of $\mathcal{S}_t$, $\{\chi^1,\ldots,\chi^n\}$ is a local coordinate chart for $\mathcal{S}_t$ and such that the unit normal vector field $\boldsymbol{\eta}_i$, for $i\in\{1,\ldots,k\}$ is tangent to the coordinate curve χ^{n+i}, for $i\in\{1,\ldots,k\}$. Note that at any point of $\mathcal{S}_t$, we have $h_{\alpha(n+i)}=\delta_{\alpha(n+i)}$, for $i\in\{1,\ldots,k\}$ and $\alpha\in\{1,\ldots,m\}$. We denote the connection coefficients for the Levi-Civita connections $\nabla^{\boldsymbol{h}}$ and $\nabla^{\boldsymbol{g}_t}$ corresponding to the metrics $\boldsymbol{h}$ and $\boldsymbol{g}_t$ by $\tilde{\gamma}^{\alpha}_{\beta\gamma}$ and γ^i_{jk}, respectively.

In order to describe the dynamics of the motion of $\mathcal{B}$, the Lagrangian field theory should be formulated with respect to the fixed space $\mathcal{Q}$. For an elastic material, the Lagrangian density $\mathcal{L}$ can be written as

$$\mathcal{L}=\mathcal{L}(\boldsymbol{X},\tilde{\varphi},\dot{\tilde{\varphi}},\tilde{\boldsymbol{F}},\boldsymbol{G},\boldsymbol{h}),$$

where $\tilde{\boldsymbol{F}}=T\tilde{\varphi}_t=\psi_{t*}\boldsymbol{F}$ and $\boldsymbol{F}=T\varphi_t$ are the deformation gradients of $\tilde{\varphi}_t$ and φ_t, respectively. We assume that the Lagrangian density can be written as

$$\mathcal{L}=\frac{1}{2}\rho_0\langle\!\langle\boldsymbol{\Upsilon},\boldsymbol{\Upsilon}\rangle\!\rangle_{\boldsymbol{h}}-\rho_0W(\boldsymbol{X},\tilde{\boldsymbol{F}},\boldsymbol{G},\boldsymbol{h}),\tag{2.1}$$

[3] We can for example take $\mathcal{S}=\mathcal{S}_{t_o}$ for some fixed time t_o.

[4] Note that following Nash [1956]'s embedding theorem, such an isometric embedding always exists for $\dim\mathcal{Q}$ large enough.

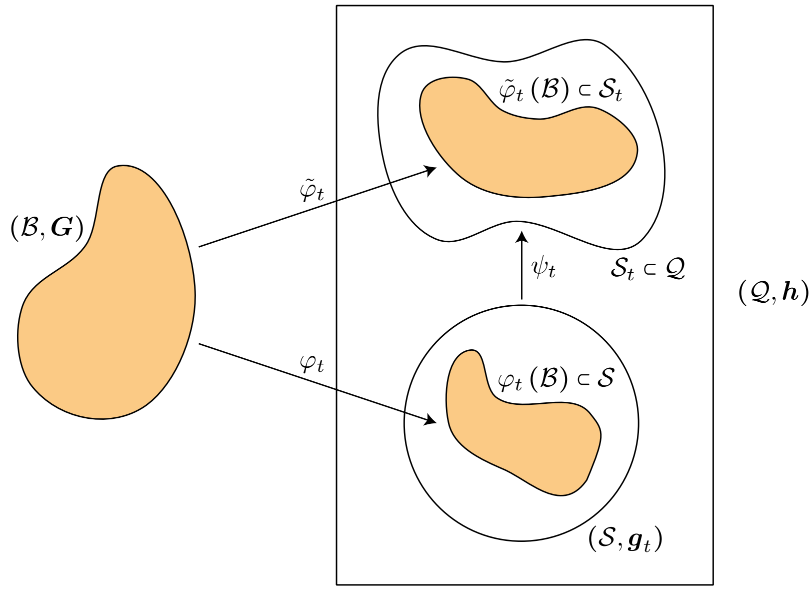

Figure 2.1: *Motion of an elastic body in an evolving ambient space.*

where ρ_0 is the material mass density, $\boldsymbol{\Upsilon} := \dot{\tilde{\varphi}} = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta} \circ \varphi_t$ is the material velocity vector field of $\tilde{\varphi}$, $\boldsymbol{V}$ is the material velocity vector field of φ, $\boldsymbol{\zeta} = \partial\psi/\partial t$ is the velocity of a given, fixed point $x \in \mathcal{S}$ as it moves in $\mathcal{Q}$, and $W = W(\boldsymbol{X}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{h})$ is the elastic energy density (energy function).

Remark 2.1. Note that since $\boldsymbol{g}_t := \psi_t^* \boldsymbol{h}$, i.e., ψ_t is an isometry between $(\mathcal{S}, \boldsymbol{g}_t)$ and $(\mathcal{S}_t, \boldsymbol{h})$, by objectivity (the isometry ψ_t can be interpreted at as a change of observer), the dependence of the elastic energy on $\tilde{\boldsymbol{F}} = \psi_{t*}\boldsymbol{F}$ reduces to a dependence on $\boldsymbol{F}$ only, and it should also depend on $\boldsymbol{G}$ and $\boldsymbol{g}_t$ (instead of $\boldsymbol{h}$) so that one can get a scalar out of $\boldsymbol{F}$. Hence, we have[5]

$$W(\boldsymbol{X}, \tilde{\boldsymbol{F}}, \boldsymbol{G}, \boldsymbol{h}) = W(\boldsymbol{X}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}_t) \,. \tag{2.2}$$

For a continuum with body forces $\boldsymbol{\beta}$ (not necessarily conservative), the Lagrange-d'Alembert principle states that [Marsden and Ratiu, 2003]

$$\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L} dV dt + \int_{t_0}^{t_1} \int_{\mathcal{B}} \rho_0 \boldsymbol{\beta}^\flat \cdot \delta\tilde{\varphi} J^{-1} dV dt = 0 \,, \tag{2.3}$$

where $^\flat$ denotes the flat operator for lowering tensor indices, $\boldsymbol{\beta}$ denotes body forces per unit deformed mass, and J denotes the Jacobien of the deformation φ_t. Note that $\boldsymbol{\beta}$ is not necessarily tangent to $\mathcal{S}_t$ and we write it as $\boldsymbol{\beta} = \boldsymbol{\beta}_\parallel + \sum_{i=1}^k b^i \boldsymbol{\eta}_i$, where $\boldsymbol{\beta}_\parallel$ is the part of $\boldsymbol{\beta}$ tangent to $\mathcal{S}_t$, and b^i, for $i \in \{1, ..., k\}$, is its components along the i^{th} normal $\boldsymbol{\eta}_i$. The action is defined on the material manifold $(\mathcal{B}, \boldsymbol{G})$ as

$$S(\tilde{\varphi}) = \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L}(\boldsymbol{X}, \tilde{\varphi}, \dot{\tilde{\varphi}}, \tilde{\boldsymbol{F}}, \boldsymbol{G}, \boldsymbol{h}) dV(\boldsymbol{X}) dt \,, \tag{2.4}$$

where $dV(\boldsymbol{X})$ is the Riemannian volume element on $\mathcal{B}$. For the assumed Lagrangian (2.1), we have $S = S_T + S_W$, where

$$S_T = \int_{t_0}^{t_1} \int_{\mathcal{B}} \frac{1}{2} \rho_0 \langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \rangle\!\rangle_{\boldsymbol{h}} \, dV(\boldsymbol{X}) dt \,,$$
$$S_W = -\int_{t_0}^{t_1} \int_{\mathcal{B}} \rho_0 W(\boldsymbol{X}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}_t) dV(\boldsymbol{X}) dt \,.$$

[5] Another way to see this is by looking at the elastic energy as a function of the right Cauchy-Green tesnor, i.e., $W = \tilde{W}(\boldsymbol{X}, \tilde{\boldsymbol{C}}, \boldsymbol{G})$. First, we see that since $\boldsymbol{g}_t := \psi_t^* \boldsymbol{h}$, then $(\psi_t \circ \varphi_t)^* \boldsymbol{h} = \varphi_t^* \psi_t^* \boldsymbol{h} = \varphi_t^* \boldsymbol{g}_t$, i.e., the right Cauchy Green tensors $\boldsymbol{C}$ of φ_t and $\tilde{\boldsymbol{C}}$ of $\tilde{\varphi}_t$ are equal. If we denote $\boldsymbol{f} := T\psi_t$, we write in components $\tilde{C}_{AB} = f^\alpha{}_a F^a{}_A f^\beta{}_b F^b{}_B h_{\alpha\beta} = F^a{}_A F^b{}_B f^\alpha{}_a f^\beta{}_b h_{\alpha\beta} = F^a{}_A F^b{}_B g_{ab} = C_{AB}$. Therefore $W = \tilde{W}(\boldsymbol{X}, \tilde{\boldsymbol{C}}, \boldsymbol{G}) = \tilde{W}(\boldsymbol{X}, \boldsymbol{C}, \boldsymbol{G})$, that is, the elastic energy does not depend on the embedding ψ_t.

In order to take variations of the action (2.4), we consider a variation field $\tilde{\varphi}_\epsilon$ of $\tilde{\varphi}$ such that $\tilde{\varphi}_0 = \tilde{\varphi}$ and define its variation as

$$\delta\tilde{\varphi} = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde{\varphi}_\epsilon \,.$$

First, we look at the resulting variations of the kinetic energy

$$\frac{d}{d\epsilon}\frac{1}{2}\left\langle\!\left\langle \boldsymbol{\Upsilon}_\epsilon, \boldsymbol{\Upsilon}_\epsilon \right\rangle\!\right\rangle_{\boldsymbol{h}} = \left\langle\!\left\langle D^{\boldsymbol{h}}_\epsilon \boldsymbol{\Upsilon}_\epsilon, \boldsymbol{\Upsilon}_\epsilon \right\rangle\!\right\rangle_{\boldsymbol{h}} \,,$$

where $D^{\boldsymbol{h}}_\epsilon$ denotes the covariant derivative along the curve $\epsilon \mapsto \tilde{\varphi}_\epsilon(X,t)$ for fixed X and t. Using the symmetry lemma, we have $D^{\boldsymbol{h}}_\epsilon \boldsymbol{\zeta}_\epsilon = D^{\boldsymbol{h}}_t \delta\tilde{\varphi}$, where $D^{\boldsymbol{h}}_t$ denotes the covariant derivative along the curve $t \mapsto \tilde{\varphi}(X,t)$ for fixed X. Therefore, we can write

$$\frac{d}{d\epsilon}\frac{1}{2}\left\langle\!\left\langle \boldsymbol{\Upsilon}_\epsilon, \boldsymbol{\Upsilon}_\epsilon \right\rangle\!\right\rangle_{\boldsymbol{h}} = \left\langle\!\left\langle D^{\boldsymbol{h}}_t \delta\tilde{\varphi}, \boldsymbol{\Upsilon} \right\rangle\!\right\rangle_{\boldsymbol{h}} = \frac{d}{dt}\left\langle\!\left\langle \delta\tilde{\varphi}, \boldsymbol{\Upsilon} \right\rangle\!\right\rangle_{\boldsymbol{h}} - \left\langle\!\left\langle \delta\tilde{\varphi}, D^{\boldsymbol{h}}_t \boldsymbol{\Upsilon} \right\rangle\!\right\rangle_{\boldsymbol{h}} \,.$$

Assuming that the variation of $\tilde{\varphi}$ is fixed at t_0 and t_1, i.e., $\delta\tilde{\varphi}(t_0) = \delta\tilde{\varphi}(t_1) = 0$, the first term on the right-hand side does not contribute to the action. We decompose the velocity $\boldsymbol{\Upsilon}$ into tangent and normal components as $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_\parallel + \boldsymbol{\Upsilon}_\perp$, where $\boldsymbol{\Upsilon}_\parallel = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta}_\parallel$ and $\boldsymbol{\Upsilon}_\perp = \sum_{i=1}^k \zeta^i \boldsymbol{\eta}_i$, such that $\boldsymbol{\zeta}$ is written in terms of its tangent and normal components as $\boldsymbol{\zeta} = \boldsymbol{\zeta}_\parallel + \sum_{i=1}^k \zeta^i \boldsymbol{\eta}_i$. We denote the acceleration in $\mathcal{Q}$ by $\boldsymbol{\Gamma} = D^{\boldsymbol{h}}_t \boldsymbol{\Upsilon}$ and decompose it into tangent and normal with respect to $\mathcal{S}_t$ as $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_\parallel + \sum_{i=1}^k \Gamma^i \boldsymbol{\eta}_i$. We denote by $\boldsymbol{A} = \psi_t^* \boldsymbol{\Gamma}_\parallel$ the intrinsic acceleration of $\mathcal{S}$. The variation of the kinetic energy is calculated as

$$\begin{aligned}
\delta\left(\frac{1}{2}\rho_0 \left\langle\!\left\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \right\rangle\!\right\rangle_{\boldsymbol{h}}\right) &= \frac{d}{dt}\left\langle\!\left\langle \delta\tilde{\varphi}, \rho_0\boldsymbol{\Upsilon} \right\rangle\!\right\rangle_{\boldsymbol{h}} - \left\langle\!\left\langle \delta\tilde{\varphi}_\parallel, \rho_0\boldsymbol{\Gamma}_\parallel \right\rangle\!\right\rangle_{\boldsymbol{h}} - \rho_0 \sum_{i=n+1}^{m} \Gamma^i \delta\tilde{\varphi}^i \\
&= \frac{d}{dt}\left\langle\!\left\langle \delta\tilde{\varphi}, \rho_0\boldsymbol{\Upsilon} \right\rangle\!\right\rangle_{\boldsymbol{h}} - \left\langle\!\left\langle \psi^*\delta\tilde{\varphi}_\parallel, \rho_0\boldsymbol{A} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} - \rho_0 \sum_{i=n+1}^{m} \Gamma^i \delta\tilde{\varphi}^i \,,
\end{aligned}$$

where $\delta\tilde{\varphi}_\parallel$ is the part of $\delta\tilde{\varphi}$ tangent to $\mathcal{S}_t$ and $\delta\tilde{\varphi}^i$ is its component along $\boldsymbol{\eta}_i$, for $i \in \{1, \dots, k\}$. Assuming that the variation of $\tilde{\varphi}$ is fixed on the boundary, i.e., $\delta\tilde{\varphi}|_{\partial\varphi(\mathcal{B})} = 0$, we obtain

$$\delta S_T = -\int_{t_0}^{t_1}\int_{\mathcal{B}}\left(\left\langle\!\left\langle \psi^*\delta\tilde{\varphi}_\parallel, \rho_0\boldsymbol{A} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \rho_0 \sum_{i,j=n+1}^{m} \Gamma^i \delta\tilde{\varphi}^j \delta_{ij}\right) dV(\boldsymbol{X})dt \,. \tag{2.5}$$

Next we compute the components of the acceleration.

Proposition 2.1. *The intrinsic (tangent) and extrinsic (normal) accelerations are*

$$\begin{aligned}
\boldsymbol{A} &= D^{\boldsymbol{g}_t}_t(\boldsymbol{V}+\boldsymbol{Z}) + 2\sum_{i=1}^{k}\zeta^i \boldsymbol{g}^\sharp \cdot \boldsymbol{k}_{(i)t}\cdot(\boldsymbol{V}+\boldsymbol{Z}) \\
&= D^{\boldsymbol{g}_t}_t(\boldsymbol{V}+\boldsymbol{Z}) + \boldsymbol{g}^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t}\cdot(\boldsymbol{V}+\boldsymbol{Z}) - \left(\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat + \left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat\right]^\mathsf{T}\right)\cdot(\boldsymbol{V}+\boldsymbol{Z}) \,,
\end{aligned} \tag{2.6a}$$

$$\Gamma^i = \frac{d\zeta^i}{dt} - \boldsymbol{k}_{(i)t}(\boldsymbol{V}+\boldsymbol{Z}, \boldsymbol{V}+\boldsymbol{Z}) \,, \quad i = 1, \dots, k \,, \tag{2.6b}$$

where $D^{\boldsymbol{g}_t}_t$ denotes the covariant derivative along the curve $t \mapsto \varphi(X,t)$ for fixed X, $\boldsymbol{Z} := \left(\psi_t^*\boldsymbol{\zeta}_\parallel\right)\circ\varphi_t$ is the tangent part of the velocity $\boldsymbol{\zeta}$, and $^\mathsf{T}$ denotes the transpose operator with respect to the metric $\boldsymbol{g}_t$. If the ambient space evolution is transversal, i.e., $\boldsymbol{Z} = \boldsymbol{0}$, the tangent and normal accelerations read

$$\boldsymbol{A} = D^{\boldsymbol{g}_t}_t\boldsymbol{V} + 2\sum_{i=1}^{k}\zeta^i \boldsymbol{g}^\sharp \cdot \boldsymbol{k}_{(i)t}\cdot\boldsymbol{V} = D^{\boldsymbol{g}_t}_t\boldsymbol{V} + \boldsymbol{g}^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t}\cdot\boldsymbol{V} \,, \tag{2.7a}$$

$$\Gamma^i = \frac{d\zeta^i}{dt} - \boldsymbol{k}_{(i)t}(\boldsymbol{V}, \boldsymbol{V}) \,, \quad i = 1, \dots, k \,. \tag{2.7b}$$

Proof: Recall that $\dim \mathcal{S}_t = n$, and $\dim \mathcal{Q} = n + k = m$. Also, recall that we take $\{\chi^\alpha\}$ to be a local coordinate chart for $\mathcal{Q}$ such that at any point of $\mathcal{S}_t$, $\{\chi^1, ..., \chi^n\}$ is a local coordinate chart for $\mathcal{S}_t$ and such that the unit normal vector field $\boldsymbol{\eta}_i$, for $i \in \{1, ..., k\}$ is tangent to the coordinate curve χ^{n+i}, for $i \in \{1, ..., k\}$. Note that at any point of $\mathcal{S}_t$, we have $h_{\alpha(n+i)} = \delta_{\alpha(n+i)}$, for $i \in \{1, ..., k\}$ and $\alpha \in \{1, ..., m\}$. In the local coordinate chart $\{\chi^\alpha\}$, the acceleration $\mathbf{\Gamma}$ in components reads

$$\Gamma^\alpha = \frac{d\Upsilon^\alpha}{dt} + \sum_{\beta,\theta=1}^{m} \tilde{\gamma}^\alpha{}_{\beta\theta} \Upsilon^\beta \Upsilon^\theta = \frac{d\Upsilon^\alpha}{dt} + \sum_{\beta,\theta=1}^{n} \tilde{\gamma}^\alpha{}_{\beta\theta} \Upsilon^\beta \Upsilon^\theta + 2 \sum_{\beta=1}^{n} \sum_{i=1}^{k} \tilde{\gamma}^\alpha{}_{\beta i} \Upsilon^\beta \Upsilon^i + \sum_{i,j=n+1}^{m} \tilde{\gamma}^\alpha{}_{ij} \Upsilon^i \Upsilon^j . \tag{2.8}$$

For $\alpha, \beta < n+1$ and $i, j \in \{1, ..., k\}$, denoting by $\boldsymbol{\kappa}_{(i)t}$ the second fundamental form of $\mathcal{S}_t$ along the normal $\boldsymbol{\eta}_i$ so that $\boldsymbol{k}_{(i)t} = \psi_t^* \boldsymbol{\kappa}_{(i)t}$, we note the following

$$\begin{aligned}
\kappa_{(i)}{}^\alpha{}_\beta &= h^{\alpha\theta} \left\langle\!\left\langle \tilde{\nabla}_{\partial_\beta} \boldsymbol{\eta}_i, \partial_\theta \right\rangle\!\right\rangle_{\boldsymbol{h}} = \tilde{\gamma}^\alpha{}_{\beta i} , \\
\kappa_{(i)\alpha\beta} &= \left\langle\!\left\langle \tilde{\nabla}_{\partial_\beta} \boldsymbol{\eta}_i, \partial_\alpha \right\rangle\!\right\rangle_{\boldsymbol{h}} = -\left\langle\!\left\langle \boldsymbol{\eta}_i, \tilde{\nabla}_{\partial_\beta} \partial_\alpha \right\rangle\!\right\rangle_{\boldsymbol{h}} = -\tilde{\gamma}^i{}_{\alpha\beta} .
\end{aligned}$$

We also have for $\alpha, \beta < n+1$ and $i, j \in \{1, ..., k\}$

$$\begin{aligned}
\tilde{\gamma}^\alpha{}_{ij} &= h^{\alpha\beta} \left\langle\!\left\langle \tilde{\nabla}_{\boldsymbol{\eta}_i} \boldsymbol{\eta}_j, \partial_\beta \right\rangle\!\right\rangle_{\boldsymbol{h}} \\
&= -h^{\alpha\beta} \left\langle\!\left\langle \boldsymbol{\eta}_j, \tilde{\nabla}_{\boldsymbol{\eta}_i} \partial_\beta \right\rangle\!\right\rangle_{\boldsymbol{h}} , \quad \text{since} \quad \left\langle\!\left\langle \boldsymbol{\eta}_j, \partial_\beta \right\rangle\!\right\rangle_{\boldsymbol{h}} = 0 , \\
&= -h^{\alpha\beta} \left\langle\!\left\langle \boldsymbol{\eta}_j, \tilde{\nabla}_{\partial_\beta} \boldsymbol{\eta}_i \right\rangle\!\right\rangle_{\boldsymbol{h}} , \quad \text{since for the coordinate basis} \quad \tilde{\nabla}_{\boldsymbol{\eta}_i} \partial_\beta = \tilde{\nabla}_{\partial_\beta} \boldsymbol{\eta}_i , \\
&= h^{\alpha\beta} \left\langle\!\left\langle \boldsymbol{\eta}_i, \tilde{\nabla}_{\partial_\beta} \boldsymbol{\eta}_j \right\rangle\!\right\rangle_{\boldsymbol{h}} , \quad \text{since} \quad \left\langle\!\left\langle \boldsymbol{\eta}_i, \boldsymbol{\eta}_j \right\rangle\!\right\rangle_{\boldsymbol{h}} = \delta_{ij} , \\
&= h^{\alpha\beta} \left\langle\!\left\langle \boldsymbol{\eta}_i, \tilde{\nabla}_{\boldsymbol{\eta}_j} \partial_\beta \right\rangle\!\right\rangle_{\boldsymbol{h}} , \quad \text{since for the coordinate basis} \quad \tilde{\nabla}_{\partial_\beta} \boldsymbol{\eta}_j = \tilde{\nabla}_{\boldsymbol{\eta}_j} \partial_\beta , \\
&= -h^{\alpha\beta} \left\langle\!\left\langle \tilde{\nabla}_{\boldsymbol{\eta}_j} \boldsymbol{\eta}_i, \partial_\beta \right\rangle\!\right\rangle_{\boldsymbol{h}} , \quad \text{since} \quad \left\langle\!\left\langle \boldsymbol{\eta}_i, \partial_\beta \right\rangle\!\right\rangle_{\boldsymbol{h}} = 0 , \\
&= -\tilde{\gamma}^\alpha{}_{ji} .
\end{aligned}$$

However, since the connection is Levi-Civita, we have $\tilde{\gamma}^\alpha{}_{ij} = \tilde{\gamma}^\alpha{}_{ji}$. Hence, it follows that $\tilde{\gamma}^\alpha{}_{ji} = 0$. Similarly, we can show that $\tilde{\gamma}^k{}_{ji} = 0$ for $i, j, k \in \{1, ..., k\}$. We also see that for $\alpha, \beta < n+1$ and $i, j \in \{1, ..., k\}$

$$\tilde{\gamma}^i{}_{\alpha j} = \left\langle\!\left\langle \boldsymbol{\eta}_j, \tilde{\nabla}_{\boldsymbol{\eta}_i} \partial_\alpha \right\rangle\!\right\rangle_{\boldsymbol{h}} = -\left\langle\!\left\langle \tilde{\nabla}_{\boldsymbol{\eta}_i} \boldsymbol{\eta}_j, \partial_\alpha \right\rangle\!\right\rangle_{\boldsymbol{h}} = -\tilde{\gamma}^\beta{}_{ji} h_{\alpha\beta} = 0 .$$

Therefore, it follows from (2.8) that

$$\begin{aligned}
\Gamma^\alpha &= \frac{d\Upsilon^\alpha}{dt} + \sum_{\beta,\theta=1}^{n} \tilde{\gamma}^\alpha{}_{\beta\theta} \Upsilon^\beta \Upsilon^\theta + 2 \sum_{\beta=1}^{n} \sum_{i=1}^{k} \Upsilon^i \kappa_{(i)}{}^\alpha{}_\beta \Upsilon^\beta , \quad \text{for} \quad \alpha < n+1 , \\
\Gamma^i &= \frac{d\Upsilon^i}{dt} - \sum_{\beta,\theta=1}^{n} \kappa_{(i)\beta\theta} \Upsilon^\beta \Upsilon^\theta , \quad i = 1, ..., k .
\end{aligned} \tag{2.9}$$

Note that for $\alpha < n$

$$\frac{d\Upsilon^\alpha}{dt} + \sum_{\beta,\theta=1}^{n-1} \tilde{\gamma}^\alpha{}_{\beta\theta} \Upsilon^\beta \Upsilon^\theta = \left(\psi_{t*} D_t^{\boldsymbol{g}_t} \psi_t^* \mathbf{\Upsilon}_\parallel \right)^\alpha ,$$

where we recall that $D_t^{\boldsymbol{g}_t}$ denotes the covariant derivative along the curve $t \mapsto \varphi(X, t)$ for fixed X. Therefore, (2.9) can be written as

$$\mathbf{\Gamma}_\parallel = \left(D_t^{\boldsymbol{h}} \mathbf{\Upsilon} \right)_\parallel = \psi_{t*} D_t^{\boldsymbol{g}_t} \psi_t^* \mathbf{\Upsilon}_\parallel + 2 \sum_{i=1}^{k} \zeta^i \boldsymbol{g}_t^\sharp \cdot \boldsymbol{\kappa}_{(i)t} \cdot \mathbf{\Upsilon}_\parallel , \tag{2.10a}$$

$$\Gamma^i = \left(D_t^{\boldsymbol{h}} \mathbf{\Upsilon} \right)^i = \frac{d\zeta^i}{dt} - \boldsymbol{\kappa}_{(i)t} \left(\mathbf{\Upsilon}_\parallel, \mathbf{\Upsilon}_\parallel \right) , \quad i = 1, ..., k . \tag{2.10b}$$

Denoting $\boldsymbol{A} = \psi_t^* \mathbf{\Gamma}_\parallel$ and $\boldsymbol{Z} := \psi_t^* \boldsymbol{\zeta}_\parallel$, (2.10a) reads

$$\boldsymbol{A} = D_t^{\boldsymbol{g}_t} \left(\boldsymbol{V} + \boldsymbol{Z} \right) + 2 \sum_{i=1}^{k} \zeta^i \boldsymbol{g}^\sharp \cdot \boldsymbol{k}_{(i)t} \cdot \left(\boldsymbol{V} + \boldsymbol{Z} \right) .$$

We prove in Lemma A.1 that $\frac{\partial \boldsymbol{g}_t}{\partial t} = 2\sum_{i=1}^{k}\zeta^i \boldsymbol{k}_{(i)t} + \mathfrak{L}_{\boldsymbol{Z}}\boldsymbol{g}_t = 2\sum_{i=1}^{k}\zeta^i \boldsymbol{k}_{(i)t} + \nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat + \left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat\right]^{\mathsf{T}}$. Hence, we rewrite (2.10) as follows

$$\boldsymbol{A} = D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) + \boldsymbol{g}^\sharp\cdot\frac{\partial \boldsymbol{g}_t}{\partial t}\cdot\left(\boldsymbol{V}+\boldsymbol{Z}\right) - \left(\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat + \left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat\right]^{\mathsf{T}}\right)\cdot\left(\boldsymbol{V}+\boldsymbol{Z}\right), \tag{2.11a}$$

$$\Gamma^i = \frac{d\zeta^i}{dt} - \boldsymbol{k}_{(i)t}\left(\boldsymbol{V}+\boldsymbol{Z},\boldsymbol{V}+\boldsymbol{Z}\right), \quad i = 1,\dots,k. \tag{2.11b}$$

If the ambient space evolution is transversal, i.e., $\boldsymbol{Z} = \boldsymbol{0}$, then the tangent and accelerations read

$$\boldsymbol{A} = D_t^{\boldsymbol{g}_t}\boldsymbol{V} + 2\sum_{i=1}^{k}\zeta^i\boldsymbol{g}^\sharp\cdot\boldsymbol{k}_{(i)t}\cdot\boldsymbol{V} = D_t^{\boldsymbol{g}_t}\boldsymbol{V} + \boldsymbol{g}^\sharp\cdot\frac{\partial \boldsymbol{g}_t}{\partial t}\cdot\boldsymbol{V}, \tag{2.12a}$$

$$\Gamma^i = \frac{d\zeta^i}{dt} - \boldsymbol{k}_{(i)t}\left(\boldsymbol{V},\boldsymbol{V}\right), \quad i = 1,\dots,k. \tag{2.12b}$$

■

The variation of the elastic energy is calculated as

$$\delta W = \frac{\partial W}{\partial \tilde{\boldsymbol{F}}}:\boldsymbol{L}_{\delta\tilde{\varphi}}\tilde{\boldsymbol{F}} + \frac{\partial W}{\partial \boldsymbol{h}}:\boldsymbol{L}_{\delta\tilde{\varphi}}\boldsymbol{h},$$

where $\boldsymbol{L}_{\boldsymbol{u}}\boldsymbol{w}$ denotes the Lie derivative of $\boldsymbol{w}$ with respect to $\boldsymbol{u}$. However, note that for an arbitrary time-independent material vector field $\boldsymbol{U}$, one has

$$\boldsymbol{L}_{\delta\tilde{\varphi}}\tilde{\boldsymbol{F}}\boldsymbol{U} = \left[\frac{d}{d\epsilon}\left(\tilde{\varphi}_\epsilon\circ\tilde{\varphi}_s^{-1}\right)^*\tilde{\boldsymbol{F}}\boldsymbol{U}\right]_{s=\epsilon} = \left[\frac{d}{d\epsilon}\tilde{\varphi}_{s*}\tilde{\varphi}_\epsilon^*\tilde{\varphi}_{\epsilon*}\boldsymbol{U}\right]_{s=\epsilon} = \left[\frac{d}{d\epsilon}\tilde{\varphi}_{s*}\boldsymbol{U}\right]_{s=\epsilon} = \boldsymbol{0}. \tag{2.13}$$

Thus, $\boldsymbol{L}_{\delta\tilde{\varphi}}\tilde{\boldsymbol{F}} = \boldsymbol{0}$. We also obtain, by using (A.10) and similarly to (A.11), that

$$\boldsymbol{L}_{\delta\tilde{\varphi}}\boldsymbol{h} = \mathfrak{L}_{\delta\tilde{\varphi}}\boldsymbol{h} = \psi_*\left(\mathfrak{L}_{\psi^*\delta\tilde{\varphi}_\parallel}\boldsymbol{g}_t + 2\sum_{i=1}^{k}\delta\tilde{\varphi}^i\boldsymbol{k}_{(i)t}\right), \tag{2.14}$$

where $\mathfrak{L}$ denotes the autonomous Lie derivative, $\delta\tilde{\varphi}_\parallel$ is the part of $\delta\tilde{\varphi}$ tangent to $\psi_t(\mathcal{S})$, and $\delta\tilde{\varphi}_n$ is its normal component. Therefore, recalling (2.2), it follows that

$$\delta W = \frac{\partial W}{\partial \boldsymbol{h}}:\left[\psi_*\left(\mathfrak{L}_{\psi^*\delta\tilde{\varphi}_\parallel}\boldsymbol{g}_t + 2\sum_{i=1}^{k}\delta\tilde{\varphi}^i\boldsymbol{k}_{(i)t}\right)\right] = \frac{\partial W}{\partial \boldsymbol{g}}:\left(\mathfrak{L}_{\psi^*\delta\tilde{\varphi}_\parallel}\boldsymbol{g}_t + 2\sum_{i=1}^{k}\delta\tilde{\varphi}^i\boldsymbol{k}_{(i)t}\right). \tag{2.15}$$

Let us first assume that the variations of $\tilde{\varphi}$ are tangent to $\mathcal{S}_t$, i.e., $\delta\tilde{\varphi}^i = 0, \forall i \in \{1,\dots,k\}$. Therefore, the variation of the action associated with the elastic energy reads

$$\delta S_W = -\int_{t_0}^{t_1}\int_{\mathcal{B}}\rho_0\frac{\partial W}{\partial \boldsymbol{g}}:\mathfrak{L}_{\psi^*\delta\tilde{\varphi}_\parallel}\boldsymbol{g}_t dV dt.$$

Note, however, that $\mathfrak{L}_{\psi^*\delta\tilde{\varphi}_\parallel}\boldsymbol{g}_t = \nabla^{\boldsymbol{g}_t}\psi^*\delta\tilde{\varphi}_\parallel^\flat + \left[\nabla^{\boldsymbol{g}_t}\psi^*\delta\tilde{\varphi}_\parallel^\flat\right]^{\mathsf{T}}$. Hence, by symmetry of $\boldsymbol{g}_t$, we can write

$$\begin{aligned}
\delta S_W &= -\int_{t_0}^{t_1}\int_{\mathcal{B}}2\rho_0\frac{\partial W}{\partial \boldsymbol{g}}:\nabla^{\boldsymbol{g}_t}\psi^*\delta\tilde{\varphi}_\parallel^\flat dV dt \\
&= -\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}2\rho\frac{\partial W}{\partial \boldsymbol{g}}:\nabla^{\boldsymbol{g}_t}\psi^*\delta\tilde{\varphi}_\parallel^\flat dv dt \\
&= -\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}2\rho\frac{\partial W}{\partial g_{ab}}\left(\psi^*\delta\tilde{\varphi}_\parallel\right)_{a|b} dv dt \\
&= -\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\operatorname{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\cdot\psi^*\delta\tilde{\varphi}_\parallel^\flat\right)dv dt + \int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle\operatorname{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\right),\psi^*\delta\tilde{\varphi}_\parallel\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} dv dt \\
&= -\int_{t_0}^{t_1}\int_{\partial\varphi_t(\mathcal{B})}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\cdot\psi^*\delta\tilde{\varphi}_\parallel^\flat\right)\cdot\mathbf{n} da dt + \int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle\operatorname{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\right),\psi^*\delta\tilde{\varphi}_\parallel\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} dv dt,
\end{aligned}$$

where ρ is the mass density in $\mathcal{S}$, $\operatorname{div}_{\boldsymbol{g}_t}$ (surface divergence) denotes the divergence operator in $(\mathcal{S},\boldsymbol{g}_t)$, and $\mathbf{n}$ is the unit normal vector to $\partial\varphi_t(\mathcal{B})$ in $\mathcal{S}$. Therefore, assuming that the variation of $\tilde{\varphi}$ is fixed on the boundary, i.e., $\delta\tilde{\varphi}|_{\partial\varphi(\mathcal{B})}=0$, we obtain

$$\delta S_W=\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle \operatorname{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\right),\psi^*\delta\tilde{\varphi}_\parallel\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} dv dt\,. \tag{2.16}$$

Hence, by (2.5) and (2.16), the Lagrange-d'Alembert principle (2.3) reads

$$\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle -\rho\boldsymbol{A}+\operatorname{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\right),\psi^*\delta\tilde{\varphi}_\parallel\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} dv dt+\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle \rho\boldsymbol{b},\psi^*\delta\tilde{\varphi}_\parallel\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} dv dt=0\,,$$

where $\boldsymbol{b}=\psi^*\boldsymbol{\beta}_\parallel$. Therefore, by arbitrariness of $\delta\tilde{\varphi}_\parallel$, we obtain the following tangent Euler-Lagrange equations

$$\operatorname{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\right)+\rho\boldsymbol{b}=\rho\boldsymbol{A}\,, \tag{2.17}$$

where, as in (2.6a), $\boldsymbol{A}=D_t^{\boldsymbol{g}_t}(\boldsymbol{V}+\boldsymbol{Z})+\boldsymbol{g}^\sharp\cdot\frac{\partial \boldsymbol{g}_t}{\partial t}\cdot(\boldsymbol{V}+\boldsymbol{Z})-\left(\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat+\left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat\right]^\mathsf{T}\right)\cdot(\boldsymbol{V}+\boldsymbol{Z})$. In terms of the Cauchy stress tensor $\boldsymbol{\sigma}=2\rho\frac{\partial W}{\partial \boldsymbol{g}}$, we have[6]

$$\operatorname{div}_{\boldsymbol{g}_t}\boldsymbol{\sigma}+\rho\boldsymbol{b}=\rho\boldsymbol{A}\,. \tag{2.18}$$

In the particular case of a transversal evolution, i.e., $\boldsymbol{Z}=\mathbf{0}$, the tangent Euler-Lagrange equations read

$$\operatorname{div}_{\boldsymbol{g}_t}\boldsymbol{\sigma}+\rho\boldsymbol{b}=\rho D_t^{\boldsymbol{g}_t}\boldsymbol{V}+\rho\boldsymbol{g}^\sharp\cdot\frac{\partial \boldsymbol{g}_t}{\partial t}\cdot\boldsymbol{V}\,. \tag{2.19}$$

Note that in this case the tangent Euler-Lagrange equations (2.19) are intrinsic, that is, they can be written in terms of the evolution of the intrinsic metric only and are independent of the extrinsic geometry and the embedding ψ_t.

Now, we assume that the variations of $\tilde{\varphi}$ are normal to $\mathcal{S}_t$, i.e., $\delta\tilde{\varphi}_\parallel=\mathbf{0}$. Using (2.5) and (2.15), we obtain from (2.3), by arbitrariness of $\delta\tilde{\varphi}^i$, the following normal Euler-Lagrange equations

$$-2\rho_0\frac{\partial W}{\partial \boldsymbol{g}}:\boldsymbol{k}_{(i)t}+\rho_0 b^i=\rho_0\Gamma^i\,,\quad i=1,...,k\,, \tag{2.20}$$

where, as in (2.6b), $\Gamma^i=\frac{d\zeta^i}{dt}-\boldsymbol{k}_{(i)t}(\boldsymbol{V}+\boldsymbol{Z},\boldsymbol{V}+\boldsymbol{Z})$. In terms of the Cauchy stress, we have

$$-\boldsymbol{\sigma}:\boldsymbol{k}_{(i)t}+\rho b^i=\rho\Gamma^i\,,\quad i=1,...,k\,. \tag{2.21}$$

Remark 2.2. Eq.(2.19) is identical to the tangential component of Scriven [1960]'s Eq. (27), However, we believe that the expression of the acceleration he wrote before Eq. (16) should be corrected to include the extra term that depends on the rate of change of the evolving metric as can bee seen in (2.19). If one neglects the inertial terms, the above equation is identical to Arroyo and DeSimone [2009]'s Eq. (4). However, it is not identical for an arbitrary motion to their Eq. (3). For them acceleration is

$$\frac{\partial \boldsymbol{V}}{\partial t}+\nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}}\boldsymbol{V}+V_nH\boldsymbol{V},$$

where $H=g^{ab}k_{ab}$. Note that their $\boldsymbol{k}$ is our $-\boldsymbol{k}_t$. When $\boldsymbol{Z}=\mathbf{0}$, their acceleration should be corrected to read (in their notation)[7]

$$\frac{\partial \boldsymbol{V}}{\partial t}+\nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}}\boldsymbol{V}-2V_n\boldsymbol{k}_t\cdot\boldsymbol{V}\,.$$

We also note that in the case of a 2D shell embedded as a hypersurface in $\mathbb{R}^3$, (2.21) is identical to the normal component of Scriven [1960]'s Eq. (27), although Scriven [1960] did not write down the expression of the extrinsic acceleration. Ignoring the inertial terms, the above equation is identical to Arroyo and DeSimone [2009]'s Eq. (5).

[6] An aternate proof of this result for the special case of a transversal embedding is given in Appendix B.

[7] We communicated with A. DeSimone and M. Arroyo and they kindly confirmed the mistake in their acceleration. In their derivation they followed the *master balance law* of Marsden and Hughes [1983, p. 129]. In Appendix B we show how this can be done and that the results are identical to those obtained using Hamilton's principle.

2.2 Conservation of Mass for Motion in an Evolving Ambient Space

Locally, conservation of mass is equivalent to

$$\rho(\boldsymbol{x},t)J(\boldsymbol{X},t)=\rho_0(\boldsymbol{X}),$$

where $J(\boldsymbol{X},t)=\sqrt{\frac{\det \boldsymbol{g}_t}{\det \boldsymbol{G}}}\det \boldsymbol{F}$, is the Jacobian of deformation mapping φ.[8] Thus

$$\frac{d}{dt}(\rho J)=0.$$

Note that

$$\dot{J}=\left(\operatorname{div}_{\boldsymbol{g}_t}\boldsymbol{v}\right)J+\frac{1}{2}J\operatorname{tr}\left(\frac{\partial \boldsymbol{g}_t}{\partial t}\right),$$

where the superposed dot denotes total time differentiation, i.e., $\dot{J}=\frac{dJ}{dt}$, and $\boldsymbol{v}=\boldsymbol{V}\circ\varphi_t^{-1}$. Therefore[9]

$$\dot{\rho}+\rho\operatorname{div}_{\boldsymbol{g}_t}\boldsymbol{v}+\frac{1}{2}\rho\operatorname{tr}\left(\frac{\partial \boldsymbol{g}_t}{\partial t}\right)=0\,. \tag{2.22}$$

Note that even if $\boldsymbol{v}=\boldsymbol{0}$, ρ is time dependent. Therefore, in the case of a 2D shell transversally embedded in $\mathbb{R}^3$—recalling Lemma A.1, which in this case yields $\frac{\partial \boldsymbol{g}_t}{\partial t}=2\zeta^n\boldsymbol{k}_t+\nabla^{\boldsymbol{g}_t}\boldsymbol{z}^\flat+\left[\nabla^{\boldsymbol{g}_t}\boldsymbol{z}^\flat\right]^{\mathsf{T}}$—(2.22) can be written as

$$\dot{\rho}+\rho\operatorname{div}_{\boldsymbol{g}_t}(\boldsymbol{v}+\boldsymbol{z})+\rho\zeta^n H=0\,, \tag{2.23}$$

where $H=\operatorname{tr}\boldsymbol{k}_t$ is twice the mean curvature. Eq. (2.23) is identical to the conservation of mass for shells appearing in [Scriven, 1960, Eq. (21)], [Marsden and Hughes, 1983, p. 92], and [Arroyo and DeSimone, 2009, Eq. (1)]. Note that, if we look at the spatial mass density form $\boldsymbol{\rho}:=\rho dv$, (2.22) reads

$$\boldsymbol{L}_{\boldsymbol{v}}\boldsymbol{\rho}=0\,. \tag{2.24}$$

Equivalently, one can write

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})}\rho dv=\int_{\varphi_t(\mathcal{U})}\boldsymbol{L}_{\boldsymbol{v}}(\rho dv)=\int_{\varphi_t(\mathcal{U})}\left[\boldsymbol{L}_{\boldsymbol{v}}\rho\ dv+\rho\boldsymbol{L}_{\boldsymbol{v}}(dv)\right]=0\,. \tag{2.25}$$

We know that

$$\boldsymbol{L}_{\boldsymbol{v}}(dv)=\mathfrak{L}_{\boldsymbol{v}}(dv)+\frac{\partial}{\partial t}(dv)=\left[\operatorname{div}\boldsymbol{v}+\frac{1}{2}\operatorname{tr}\left(\frac{\partial \boldsymbol{g}_t}{\partial t}\right)\right]dv. \tag{2.26}$$

Substituting (2.26) into (2.25) and localizing gives (2.22), which is the local form of conservation of mass.

2.3 Energy Balance in Nonlinear Elasticity in an Evolving Ambient Space

Let us consider an elastic body deforming in an evolving ambient space. We are interested in making an explicit connection between deformation of the elastic body embedded in this ambient space and that in an ambient space with a dynamic metric. Let the ambient space $\mathcal{S}$ move in a larger (fixed) manifold $\mathcal{Q}$, i.e. $\psi_t:\mathcal{S}\to\mathcal{Q}$. The fixed background metric in $(\mathcal{Q},\boldsymbol{h})$ induces a time-dependent metric on $\mathcal{S}$, i.e. $\boldsymbol{g}_t=\psi_t^*\boldsymbol{h}$. Energy balance can be written on $\mathcal{Q}$ but we are interested to see how it is written for an observer in $\mathcal{S}$. When the metric $\boldsymbol{g}$ of $\mathcal{S}$ is fixed, the standard material balance of energy for a given nice subset $\mathcal{U}\subset\mathcal{B}$ reads

$$\frac{d}{dt}\int_{\mathcal{U}}\rho_0\left(E+\frac{1}{2}\langle\!\langle\boldsymbol{V},\boldsymbol{V}\rangle\!\rangle_{\boldsymbol{g}}\right)dV=\int_{\mathcal{U}}\rho_0\left(\langle\!\langle\boldsymbol{B},\boldsymbol{V}\rangle\!\rangle_{\boldsymbol{g}}+R\right)dV+\int_{\partial\mathcal{U}}\left(\langle\!\langle\boldsymbol{T},\boldsymbol{V}\rangle\!\rangle_{\boldsymbol{g}}+H\right)dA\,, \tag{2.27}$$

where E, R and H are the internal energy function per unit mass, the heat supply per unit mass and the heat flux per unit area, respectively. $E=E(X,\mathsf{N},\boldsymbol{F},\boldsymbol{G},\boldsymbol{g})$, where N is the specific entropy. However, here the ambient space is evolving in time and the energy balance should be modified to accomodate this time dependence. First, let us look at an example to motivate our discussion.

[8] Note that the Jacobian of the deformation $\tilde{\varphi}$ is equal to that of φ, i.e., $\sqrt{\frac{\det \boldsymbol{h}}{\det \boldsymbol{G}}}\det\tilde{\boldsymbol{F}}=\sqrt{\frac{\det \boldsymbol{g}_t}{\det \boldsymbol{G}}}\det \boldsymbol{F}$, which follows from $\boldsymbol{g}_t:=\psi_t^*\boldsymbol{h}$.

[9] Note that there is a typo in a similar equation in [Marsden and Hughes, 1983, p. 92].

Example 2.1. Suppose the ambient space is a 2-dimensional sphere of radius R that shrinks/expands in time. Then, whatever elastic material lives on this sphere will be compressed/stretched over time. As a simple case, assume that the material manifold is also a sphere, with radius equal to the initial radius of the ambient sphere. Assume that the deformation map φ is constant over time, as the metric evolves. This means that there will be an increase in elastic energy over time, not accounted for in terms of the work done by external forces–since there are no external forces.

Let the ambient metric, as a function of time be $g_{ij}(\theta,\phi,t) = f(t) g_{ij}^{\text{sphere}}(R)(\theta,\phi)$, where t is time, $f(t)$ is some function of time (the shrinkage/expansion factor) such that $f(t)>0$, $f(t_0)=1$, and $\boldsymbol{g}_t{}^{\text{sphere}}(R)$ is the metric of the 2-sphere with radius R. Note that this is a uniform rescaling of the metric. Then, let the material manifold be just $G_{IJ}(\Theta,\Phi) = G_{IJ}^{\text{sphere}}(R)(\Theta,\Phi)$, and let the deformation map simply send Θ to θ and Φ to ϕ at all times. Therefore, even though the material "is not moving" in terms of the coordinates ϕ and θ (a given material point sits at the same ϕ and θ at all times), it is shrinking/expanding. Note that $\Psi = \Psi(\boldsymbol{X},\boldsymbol{C})$, where $C_{AB} = F^a{}_A F^b{}_B g_{ab} f(t)$. Thus, even if $F^a{}_B = \delta^a{}_A$, we have $C_{AB} = \delta^a{}_A \delta^b{}_B g_{ab} f(t)$. This means that Ψ explicitly depends on $f(t)$ and hence there is stored elastic energy coming from the changes in the ambient space metric.

To visualize the time dependency of the metric of the ambient space, let us embed the initial sphere of radius $r=R$ in the Euclidean space $\mathbb{R}^3$. We then assume that the ambient space moves in the Euclidean space, i.e. there is a map $\psi_t : S^2(r) \to \mathbb{R}^3$. Explicitly this can be written in the spherical coordinates as $(\tilde{r},\tilde{\theta},\tilde{\phi}) = \psi_t(r,\theta,\phi) = (k(t)r,\theta,\phi)$ with $k(t)>0$. Note that deformation mapping is the inclusion map, i.e. $(\theta,\phi) = \varphi_t(\Theta,\Phi) = (\Theta,\Phi)$. Metric of the Euclidean space in spherical coordinates reads $\boldsymbol{h} = \operatorname{diag}\left(1,\tilde{r}^2,\tilde{r}^2\sin^2\theta\right)$. Now the map ψ_t induces a metric $\boldsymbol{g}_t = \psi_t^*\boldsymbol{h}$ on the ambient space that has the following representation: $\boldsymbol{g}_t = \operatorname{diag}\left(k(t)^2 r^2, k(t)^2 r^2 \sin^2\theta\right)$. It is seen that $f(t) = k(t)^2$, i.e. expanding the ambient space by k all the square distances in the ambient space with the time-dependent metric are multiplied by $f = k^2$ as expected. It is seen that time dependency of the ambient space metric can be visualized using a time-dependent embedding in a larger space with a fixed background metric (see Yavari [2010] for a similar discussion). In the sequel we will look at this in the general case of an arbitrary deformable body.

Next, we prove the following proposition:

Proposition 2.2. *Energy balance for a deformable body moving in an evolving ambient space is given by*

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\!\langle \boldsymbol{V},\boldsymbol{V}\rangle\!\rangle_{\boldsymbol{g}_t}\right) dV = \int_{\mathcal{U}} \rho_0 \left[\langle\!\langle \boldsymbol{B}+\boldsymbol{F}_{\mathit{fic}},\boldsymbol{V}\rangle\!\rangle_{\boldsymbol{g}_t} + \left(\frac{\partial E}{\partial \boldsymbol{g}} : \frac{\partial \boldsymbol{g}_t}{\partial t} + \frac{1}{2}\langle\!\langle \boldsymbol{V},\boldsymbol{V}\rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}\right) + R\right] dV + \int_{\partial\mathcal{U}} \left(\langle\!\langle \boldsymbol{T},\boldsymbol{V}\rangle\!\rangle_{\boldsymbol{g}_t} + H\right) dA, \tag{2.28}$$

where $E = E(\boldsymbol{X},\mathsf{N},\boldsymbol{F},\boldsymbol{G},\boldsymbol{g}_t)$ is the material internal energy density per unit undeformed mass, N, $\boldsymbol{B}$, $\boldsymbol{T}$, R, and H are specific entropy per unit undeformed mass, tangent body force per unit undeformed mass,[10] *traction vector, heat supply per unit undeformed mass, and heat flux per unit undeformed area, respectively. We recall that $\boldsymbol{V}$ is the velocity of φ_t and $\boldsymbol{Z} = \psi_t^*\boldsymbol{\zeta}_{\parallel}$ is the tangent velocity of the embedding ψ_t. $\boldsymbol{F}_{\mathit{fic}}$ denotes the fictitious body force due to the evolution of $\mathcal{S}_t$ and reads*

$$\begin{aligned} \boldsymbol{F}_{\mathit{fic}} &= -\left(\boldsymbol{A} - D_t^{\boldsymbol{g}_t}\boldsymbol{V}\right) = -D_t^{\boldsymbol{g}_t}\boldsymbol{Z} - 2\sum_{i=1}^{k}\zeta^i \boldsymbol{g}^\sharp\cdot\boldsymbol{k}_{(i)t}\cdot(\boldsymbol{V}+\boldsymbol{Z}) \\ &= -D_t^{\boldsymbol{g}_t}\boldsymbol{Z} - \boldsymbol{g}^\sharp\cdot\frac{\partial \boldsymbol{g}_t}{\partial t}\cdot(\boldsymbol{V}+\boldsymbol{Z}) + \left(\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat + \left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat\right]^{\mathsf{T}}\right)\cdot(\boldsymbol{V}+\boldsymbol{Z}) . \end{aligned} \tag{2.29}$$

Note that in the particular case of a transversal evolution of the ambient space, i.e., $\boldsymbol{Z}=\boldsymbol{0}$, the fictitious body force is intrinsic and reduces to

$$\boldsymbol{F}_{\mathit{fic}} = -2\sum_{i=1}^{k}\zeta^i \boldsymbol{g}^\sharp\cdot\boldsymbol{k}_{(i)t}\cdot\boldsymbol{V} = -\boldsymbol{g}^\sharp\cdot\frac{\partial \boldsymbol{g}_t}{\partial t}\cdot\boldsymbol{V} . \tag{2.30}$$

Proof: For an observer in $\mathcal{Q}$, energy balance for a sub-body $\mathcal{U}\subset\mathcal{B}$ is written as

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\!\langle \boldsymbol{\Upsilon},\boldsymbol{\Upsilon}\rangle\!\rangle_{\boldsymbol{h}}\right) dV = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \tilde{\boldsymbol{\beta}},\boldsymbol{\Upsilon}\rangle\!\rangle_{\boldsymbol{h}} + R\right) dV + \int_{\partial\mathcal{U}} \left(\langle\!\langle \tilde{\boldsymbol{T}},\boldsymbol{\Upsilon}\rangle\!\rangle_{\boldsymbol{h}} + H\right) dA,$$

[10] Note that $\boldsymbol{B} = J^{-1}\boldsymbol{b}$.

where $E = E(\boldsymbol{X}, \mathsf{N}, \tilde{\boldsymbol{F}}, \boldsymbol{G}, \boldsymbol{h})$[11] is the material internal energy density per unit undeformed mass, N, $\boldsymbol{\beta}$, $\tilde{\boldsymbol{T}}$, R, and H are specific entropy per unit undeformed mass, body force per unit undeformed mass,[12] traction vector, heat supply per unit undeformed mass, and heat flux per unit undeformed area, respectively. Body force can be decomposed into tangent and normal components with respect to $\mathcal{S}_t$ as $\boldsymbol{\beta} = \boldsymbol{\beta}_\parallel + \sum_{i=1}^k B^i \boldsymbol{\eta}_i$. Note that the traction vector is tangent to $\mathcal{S}_t$. We denote $\boldsymbol{B} = \psi_t^* \boldsymbol{\beta}_\parallel$, and $\boldsymbol{T} = \psi_t^* \tilde{\boldsymbol{T}}$. Recalling that $\boldsymbol{\Upsilon} = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta} \circ \varphi_t$ and $\boldsymbol{Z} = \psi_t^* \boldsymbol{\zeta}_\parallel$, the energy balance can be simplified to read

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \rangle\!\rangle_{\boldsymbol{h}}\right) dV = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \boldsymbol{B}, \boldsymbol{V} + \boldsymbol{Z} \rangle\!\rangle_{\boldsymbol{g}_t} + \sum_{i=1}^k B^i \zeta^i + R\right) dV + \int_{\partial\mathcal{U}} \left(\langle\!\langle \boldsymbol{T}, \boldsymbol{V} + \boldsymbol{Z} \rangle\!\rangle_{\boldsymbol{g}_t} + H\right) dA. \quad (2.31)$$

Note that

$$\frac{d}{dt}\frac{1}{2}\langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \rangle\!\rangle_{\boldsymbol{h}} = \left\langle\!\!\left\langle D_t^{\boldsymbol{h}} \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \right\rangle\!\!\right\rangle_{\boldsymbol{h}} = \langle\!\langle \boldsymbol{\Gamma}, \boldsymbol{\Upsilon} \rangle\!\rangle_{\boldsymbol{h}} = \langle\!\langle \boldsymbol{A}, \boldsymbol{V} + \boldsymbol{Z} \rangle\!\rangle_{\boldsymbol{g}_t} + \sum_{i=1}^k \zeta^i \Gamma^i, \quad (2.32)$$

and

$$\frac{d}{dt} E = \boldsymbol{L}_{\boldsymbol{\Upsilon}} E = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \tilde{\boldsymbol{F}}} : \boldsymbol{L}_{\boldsymbol{\Upsilon}} \tilde{\boldsymbol{F}} + \frac{\partial E}{\partial \boldsymbol{h}} : \boldsymbol{L}_{\boldsymbol{\Upsilon}} \boldsymbol{h}.$$

Similarly to (2.13), we see that $\boldsymbol{L}_{\boldsymbol{\Upsilon}} \tilde{\boldsymbol{F}} = \boldsymbol{0}$. Note that[13]

$$\boldsymbol{L}_{\boldsymbol{\Upsilon}} \boldsymbol{h} = \boldsymbol{L}_{\psi_* \boldsymbol{V}} \boldsymbol{h} + \boldsymbol{L}_{\boldsymbol{\zeta}} \boldsymbol{h} = \mathfrak{L}_{\psi_* \boldsymbol{V}} \boldsymbol{h} + \psi_{t*} \frac{\partial \boldsymbol{g}_t}{\partial t} = \psi_* \left(\mathfrak{L}_{(\boldsymbol{V}+\boldsymbol{Z})} \boldsymbol{g}_t + \frac{\partial \boldsymbol{g}_t}{\partial t}\right) = \psi_* \boldsymbol{L}_{\boldsymbol{V}} \boldsymbol{g}_t,$$

where we used (A.9) to write $\boldsymbol{L}_{\boldsymbol{\zeta}} \boldsymbol{h} = \psi_{t*} \frac{\partial \boldsymbol{g}_t}{\partial t}$. Therefore, it follows that in $\mathcal{Q}$

$$\frac{dE}{dt} = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \boldsymbol{h}} : \boldsymbol{L}_{\boldsymbol{\Upsilon}} \boldsymbol{h} = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \boldsymbol{g}} : \boldsymbol{L}_{\boldsymbol{V}} \boldsymbol{g}_t. \quad (2.33)$$

An observer in $\mathcal{S}$ writes the energy balance as

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}_t}\right) dV = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \boldsymbol{B}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}_t} + \Xi + R\right) dV + \int_{\partial\mathcal{U}} \left(\langle\!\langle \boldsymbol{T}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}_t} + H\right) dA, \quad (2.34)$$

where $\Xi = 0$ if the ambient space metric is fixed. Note that in $(\mathcal{S}, \boldsymbol{g}_t)$, we have

$$\frac{d}{dt}\frac{1}{2}\langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}_t} = \left\langle\!\!\left\langle D_t^{\boldsymbol{g}_t} \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} + \frac{1}{2}\langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}, \quad \text{and} \quad \frac{dE}{dt} = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \boldsymbol{g}} : \boldsymbol{L}_{\boldsymbol{V}} \boldsymbol{g}_t. \quad (2.35)$$

Let us now find Ξ. Subtracting (2.34) from (2.31) and using (2.32), (2.33), and (2.35), we obtain

$$\begin{aligned}\int_{\mathcal{U}} &\left(\rho_0 \langle\!\langle \boldsymbol{A}, \boldsymbol{Z} \rangle\!\rangle_{\boldsymbol{g}_t} + \rho_0 \left\langle\!\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t} \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} - \frac{1}{2}\rho_0 \langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} + \rho_0 \sum_{i=1}^k \zeta^i \Gamma^i\right) dV \\ &= \int_{\mathcal{U}} \left(\rho_0 \langle\!\langle \boldsymbol{B}, \boldsymbol{Z} \rangle\!\rangle_{\boldsymbol{g}_t} + \rho_0 \sum_{i=1}^k B^i \zeta^i - \Xi\right) dV + \int_{\partial\mathcal{U}} \langle\!\langle \boldsymbol{T}, \boldsymbol{Z} \rangle\!\rangle_{\boldsymbol{g}_t} dA.\end{aligned}$$

Note that $2\rho \dfrac{\partial E}{\partial \boldsymbol{g}} = 2\rho \dfrac{\partial W}{\partial \boldsymbol{g}} = \boldsymbol{\sigma}$, and $\dfrac{\partial E}{\partial \boldsymbol{g}} : \boldsymbol{L}_{\boldsymbol{Z}} \boldsymbol{g}_t = 2\dfrac{\partial E}{\partial \boldsymbol{g}} : \nabla^{\boldsymbol{g}_t} \boldsymbol{Z}$. Therefore, by using (2.17) and (2.20) we have

$$\int_{\mathcal{U}} \left(\rho_0 \left\langle\!\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t} \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} - \frac{1}{2}\rho_0 \langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}\right) dV = \int_{\partial\mathcal{U}} \langle\!\langle \boldsymbol{T}, \boldsymbol{Z} \rangle\!\rangle_{\boldsymbol{g}_t} dA + \int_{\mathcal{U}} \left(\sum_{i=1}^k \zeta^i \boldsymbol{\sigma} : \boldsymbol{k}_{(i)t} - J \left\langle\!\!\left\langle \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{\sigma}, \boldsymbol{Z} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} - \Xi\right) dV.$$

[11] Similar to the discussion of Remark 2.1, we can conclude that $E(\boldsymbol{X}, \mathsf{N}, \psi_* \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{h}) = E(\boldsymbol{X}, \mathsf{N}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}_t)$.
[12] Note that $\tilde{\boldsymbol{\beta}} = J^{-1}\boldsymbol{\beta}$.
[13] An alternate proof for this result can be found in [Marsden and Hughes, 1983, p. 101].

Also, note that

$$\begin{aligned}\int_{\partial\mathcal{U}} \langle\!\langle \boldsymbol{T}, \boldsymbol{Z} \rangle\!\rangle_{\boldsymbol{g}_t} dA &= \int_{\partial\varphi_t(\mathcal{U})} \langle\!\langle \boldsymbol{\sigma}\cdot\boldsymbol{\eta}, \boldsymbol{Z} \rangle\!\rangle_{\boldsymbol{g}_t} da \\ &= \int_{\varphi_t(\mathcal{U})} \operatorname{div}_{\boldsymbol{g}_t} \langle\!\langle \boldsymbol{\sigma}, \boldsymbol{Z} \rangle\!\rangle_{\boldsymbol{g}_t} dv \\ &= \int_{\varphi_t(\mathcal{U})} \left(\boldsymbol{\sigma} : \nabla^{\boldsymbol{g}_t} \boldsymbol{Z} + \left\langle\!\!\left\langle \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{\sigma}, \boldsymbol{Z} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} \right) dv \\ &= \int_{\mathcal{U}} \left(\frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{L}_{\boldsymbol{Z}} \boldsymbol{g}_t + J \left\langle\!\!\left\langle \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{\sigma}, \boldsymbol{Z} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} \right) dV \,.\end{aligned}$$

Thus

$$\int_{\mathcal{U}} \left(\rho_0 \left\langle\!\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t} \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} - \frac{1}{2} \rho_0 \langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} \right) dV = \int_{\mathcal{U}} \left(\frac{1}{2} \boldsymbol{\sigma} : \left(\boldsymbol{L}_{\boldsymbol{Z}} \boldsymbol{g}_t + 2 \sum_{i=1}^{k} \zeta^i \boldsymbol{k}_{(i)t} \right) - \Xi \right) dV \,.$$

However, since $\boldsymbol{\sigma} = 2\rho \frac{\partial W}{\partial \boldsymbol{g}} = 2\rho \frac{\partial E}{\partial \boldsymbol{g}}$, and $\frac{\partial \boldsymbol{g}_t}{\partial t} = 2 \sum_{i=1}^{k} \zeta^i \boldsymbol{k}_{(i)t} + \mathfrak{L}_{\boldsymbol{Z}} \boldsymbol{g}_t$, it follows that

$$\Xi = \frac{\partial E}{\partial \boldsymbol{g}} : \frac{\partial \boldsymbol{g}_t}{\partial t} - \rho_0 \left\langle\!\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t} \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} + \frac{1}{2} \rho_0 \langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} \,.$$

Therefore, the balance of energy in $(\mathcal{S}, \boldsymbol{g}_t)$ reads

$$\frac{d}{dt} \int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2} \langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}_t} \right) dV = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \boldsymbol{B} + \boldsymbol{F}_{\text{fic}}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}_t} + \frac{\partial E}{\partial \boldsymbol{g}} : \frac{\partial \boldsymbol{g}_t}{\partial t} + \frac{1}{2} \langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} + R \right) dV + \int_{\partial\mathcal{U}} \left(\langle\!\langle \boldsymbol{T}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}_t} + H \right) dA \,,$$

where the fictitious body force $\boldsymbol{F}_{\text{fic}}$ reads

$$\begin{aligned}\boldsymbol{F}_{\text{fic}} &= -\left(\boldsymbol{A} - D_t^{\boldsymbol{g}_t} \boldsymbol{V} \right) = -D_t^{\boldsymbol{g}_t} \boldsymbol{Z} - 2 \sum_{i=1}^{k} \zeta^i \boldsymbol{g}^\sharp \cdot \boldsymbol{k}_{(i)t} \cdot (\boldsymbol{V} + \boldsymbol{Z}) \\ &= -D_t^{\boldsymbol{g}_t} \boldsymbol{Z} - \boldsymbol{g}^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot (\boldsymbol{V} + \boldsymbol{Z}) + \left(\nabla^{\boldsymbol{g}_t} \boldsymbol{Z}^\flat + \left[\nabla^{\boldsymbol{g}_t} \boldsymbol{Z}^\flat \right]^{\mathsf{T}} \right) \cdot (\boldsymbol{V} + \boldsymbol{Z}) \,.\end{aligned}$$

Now, when the evolution of the ambient space is transversal, i.e., $\boldsymbol{Z} = \boldsymbol{0}$, the fictitious body force reduces to

$$\boldsymbol{F}_{\text{fic}} = -2 \sum_{i=1}^{k} \zeta^i \boldsymbol{g}^\sharp \cdot \boldsymbol{k}_{(i)t} \cdot (\boldsymbol{V} + \boldsymbol{Z}) = -\boldsymbol{g}^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{V} \,.$$

Therefore, the balance of energy in the case of a transversal evolution is intrinsic, in the sense that it only depends on the time evolution of the intrinsic metric. ∎

Remark 2.3. Note that the non-classical extra terms appearing in the energy balance (2.28) can be written as

$$\frac{\partial E}{\partial \boldsymbol{g}} : \frac{\partial \boldsymbol{g}_t}{\partial t} + \frac{1}{2} \langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} = \frac{\partial}{\partial \boldsymbol{g}} \left(E + \frac{1}{2} \langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}_t} \right) : \frac{\partial \boldsymbol{g}_t}{\partial t} \,,$$

so that this term cancels out the contribution of the rate of evolution of the energy (internal + kinetic) due to the evolving ambient metric appearing on the left hand side of (2.28).

3 Quasi-Static Deformations of the Ambient Space Metric

Let us consider a spatial metric that depends on a position-dependent parameter $\omega(\boldsymbol{x})$, e.g. $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x}, \omega(\boldsymbol{x}))$. In other words, given an initial metric, we quasi-statically deform the ambient space manifold. As an application, we can think of a situation when a thin sheet of metal is compressed between two identical and evolving surfaces to make different curved sheets, e.g. some pieces of an automobile body.

As an example, one can start with a rescaling of the spatial metric, i.e.

$$\widetilde{\boldsymbol{g}}(\boldsymbol{x}, s(\boldsymbol{x})) = e^{2\omega(\boldsymbol{x})} \boldsymbol{g}(\boldsymbol{x}).$$

Note that Jacobian of deformation in the new ambient space $\widetilde{J}$ is related to the Jacobian with respect to the old ambient space J as follows:

$$\widetilde{J} = \sqrt{\frac{\det \widetilde{\boldsymbol{g}}}{\det \boldsymbol{g}}} \det \boldsymbol{F} = e^{\frac{N\omega(\boldsymbol{x})}{2}} J,$$

where it is assumed that $\dim \mathcal{S} = N$. Having an equilibrium configuration, replacing $\boldsymbol{g}$ by its rescaled version, the equilibrium configuration will change, in general. The following two examples show the effect of a rescaling of the spatial metric on equilibrium configuration and the corresponding stresses.

3.1 Examples of elastic bodies in evolving ambient spaces and the induced stresses

We consider two examples in this section. In the first example a disk of radius R_o is embedded in a two-dimensional ambient space that is initially flat. We first show that the disk remains stress-free when the metric of the ambient space is rescaled homogeneously then we calculate the stresses in the case of a spatial metric inclusion. In the second example, we consider a spherical cap embedded in a spherical ambient space. We uniformly rescale the spatial spherical metric (equivalently changing the radius of the sphere) and calculate the resulting stresses. In both examples we assume an incompressible and isotropic solid. For such solids energy function depends on the first and second principal invariants of the right Cauchy-Green strain $\mathbf{C}$ (or the left Cauchy-Green strain $\mathbf{b}$, also known as the Finger deformation tensor), i.e. $W = W(I_1, I_2)$ [Ogden, 1997]. Note that for an incompressible solid $I_3 = J^2 = 1$. The Finger deformation tensor $\boldsymbol{b}$ has components $b^{ab} = F^a{}_A F^b{}_B G^{AB}$.[14] For an incompressible isotropic solid the Cauchy stress has the following representation [Simo and Marsden, 1984a]

$$\boldsymbol{\sigma} = \left(-p + 2I_2 \frac{\partial W}{\partial I_2}\right) \boldsymbol{g}^\sharp + 2\frac{\partial W}{\partial I_1} \boldsymbol{b}^\sharp - 2\frac{\partial W}{\partial I_2} \boldsymbol{b}^{-1}, \tag{3.1}$$

where p is the Lagrange multiplier corresponding to the incompressibility constraint $J = 1$. Note that $\boldsymbol{f}^{-1} = \boldsymbol{c}$ with components $c^{ab} = \left(F^{-1}\right)^A{}_m \left(F^{-1}\right)^B{}_n \, G_{AB} \, g^{am} g^{bn}$.

Example 3.1. (Disk in a flat 2D plane) Let us consider a two-dimensional disk of initial radius R_o made of an incompressible and isotropic material in an initially flat two-dimensional spatial manifold. We would like to calculate the stresses that occur in the new equilibrium configuration after we change the spatial metric in a radially symmetric way. In spatial polar coordinates (r, θ), the spatial metric is assumed to be:

$$\boldsymbol{g} = \begin{pmatrix} e^{2\omega(r)} & 0 \\ 0 & r^2 e^{2\omega(r)} \end{pmatrix}.$$

The nonzero connection coefficients for $\boldsymbol{g}$ are:

$$\gamma^r_{rr} = \omega'(r), \quad \gamma^r_{\theta\theta} = -r(1 + r\omega'(r)), \quad \gamma^\theta_{\theta r} = \frac{1}{r} + \omega'(r). \tag{3.2}$$

In material polar coordinates (R, Θ), the material metric reads

$$\boldsymbol{G} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}.$$

We look for solutions of the form $\varphi(R, \Theta) = (r, \theta) = (r(R), \Theta)$. Thus, $\boldsymbol{F} = \operatorname{diag}(r'(R), 1)$. This gives the Jacobian as

$$J = e^{2\omega(r)} \frac{r'(R) r}{R}.$$

Therefore, incompressibility $(J = 1)$ gives the following ODE

$$r(R) r'(R) e^{2\omega(r(R))} = R. \tag{3.3}$$

[14] The Finger deformation tensor $\boldsymbol{b}$ should not be confused with the tangent body forces per unit deformed mass. As a matter of fact, there should not be any risk of confusion in this section, since we assume that the tangent body forces are zero. Note, however, that the extrinsic body force components is denoted by b^n.

To fix rigid body translations we assume that $r(0) = 0$. Hence, $r(R)$ satisfies the following integral equation.

$$r(R) = \left(\int_0^R 2\xi e^{-2\omega(r(\xi))} d\xi \right)^{1/2} . \tag{3.4}$$

Note that the Finger tensor is

$$\boldsymbol{b} = \begin{pmatrix} \frac{e^{-4\omega}R^2}{r^2} & 0 \\ 0 & \frac{1}{R^2} \end{pmatrix} ,$$

and hence $I_1 = \frac{r^2 e^{2\omega}}{R^2} + \frac{R^2 e^{-2\omega}}{r^2}$ and $I_2 = 1$. Therefore, we obtain from (3.1) the nonzero stress components as

$$\sigma^{rr} = -e^{-2\omega}p + \frac{e^{-4\omega}R^2}{r^2}\alpha + \left(e^{-2\omega} - \frac{r^2}{R^2}\right)\beta , \quad \text{and} \quad \sigma^{\theta\theta} = -\frac{e^{-2\omega}}{r^2}p + \frac{1}{R^2}\alpha + \frac{e^{-2\omega}}{r^2}\left(1 - \frac{e^{-2\omega}R^2}{r^2}\right)\beta , \tag{3.5}$$

where $p(R)$ is the unknown Lagrange multiplier and

$$\alpha(R) = 2\frac{\partial W(I_1, I_2)}{\partial I_1} , \quad \beta(R) = 2\frac{\partial W(I_1, I_2)}{\partial I_2} .$$

In terms of the Cauchy stress tensor, the only non-trivial equilibrium equation is $\sigma^{ra}{}_{|a} = 0$, which, by using (3.2), reads

$$\frac{r(R)e^{2\omega(r(R))}}{R}\sigma^{rr}{}_{,R} + \left(\frac{1}{r} + 3\omega'\right)\sigma^{rr} - r\left(1 + r\omega'\right)\sigma^{\theta\theta} = 0 . \tag{3.6}$$

By using (3.5), the equilibrium equation (3.6) reduces to

$$p' = \beta' - \frac{R^3 e^{-4\omega}(\alpha - \beta)(r\omega' + 1)}{r^4} + \frac{r^2 e^{2\omega}(2\beta - R\beta')}{R^3} + \frac{R e^{-2\omega}(R\alpha' + 2\alpha)}{r^2} - \frac{(\alpha + 3\beta)(r\omega' + 1)}{R} . \tag{3.7}$$

Uniform scaling of the metric. Let us consider the particular case of a uniform (homogeneous) scaling of the ambient metric, i.e., $\omega(r) = \omega_o$. It follows from (3.9) that $r(R) = Re^{-\omega_o}$, and hence we get from (3.7) that $p' = \alpha'$. Assuming zero-traction at the boundary, i.e., $\sigma^{rr}(R_o) = 0$, we find that $p = \alpha$. Thus, it follows that $\sigma^{rr}(R) = 0$ and $\sigma^{\theta\theta}(R) = 0$, i.e., the disk remains stress-free when the ambient space is uniformly expanded/contracted.

Remark 3.1. Since the components of the deformation gradient do not have a coordinate-independent meaning, in a given pair of bases in the material and spatial spaces, $\boldsymbol{F}$ can be the identity matrix, but that does not mean that there is no stretch in the material. Likewise, $\boldsymbol{F}$ can be a position-dependent, non-identity matrix, and that does not mean that there is any stretch. What matters is a coordinate-independent measure of stretch. One such measure is $\boldsymbol{G} - \varphi^*\boldsymbol{g}$. If this vanishes, we can claim that there is no stretch (strain), according to this particular (and appropriate for shell-like situations) definition of strain.

In this example, we chose to represent the stretch in the ambient space (assuming uniform expansion) as a change in $\boldsymbol{g}$. The new equilibrium configuration map was given by a "rescaled" version of the original configuration map (by "rescaled" we mean rescaled in the Euclidean coordinate representation), and the final state was still stress (and strain) free. $\boldsymbol{G} - \varphi^*\boldsymbol{g}$ will still be zero, with both φ and $\boldsymbol{g}$ having changed during the process. In short, as $\boldsymbol{g}$ changes, as a result the equilibrium mapping φ changes, and likewise $\boldsymbol{F}$ changes, but the pull-back of $\boldsymbol{g}$ stays the same for the set of equilibrium configurations during the process.

A spatial metric inclusion. We assume that the spatial metric is uniformly scaled inside a disk of radius r_i and is left unchanged otherwise, i.e.

$$\omega(r) = \begin{cases} \omega_o & , \quad \text{if} \quad 0 \le r \le r_i , \\ 0 & , \quad \text{if} \quad r_i < r . \end{cases} \tag{3.8}$$

Motivated by Eshelby's inclsion problem [Eshelby, 1957] (see [Yavari and Goriely, 2013] for a discussion on finite eigenstrains and a brief history of the problem), we call this a *metric inclusion*.

It follows from (3.4) that

$$r(R)=\begin{cases} Re^{-\omega_o} & , \quad 0\le R\le r_i e^{\omega_o}\,, \\ \left[R^2+r_i^2\left(1-e^{2\omega_o}\right)\right]^{1/2} & , \quad r_i e^{\omega_o}<R\le R_o\,. \end{cases} \tag{3.9}$$

Remark 3.2. Note that if $R_o\le r_i e^{\omega_o}$, we have $r(R)=Re^{-\omega_o}$. Hence, it follows, similarly to the previous case where the ambient metric is uniformly scaled, that the disk remains stress-free. We assume in the remainder of this example that $R_o>r_i e^{\omega_o}$.

By substituting (3.8) and (3.9) into (3.7), we find

$$p'(R)=\begin{cases} \alpha'(R) & , \quad 0\le R\le r_i e^{\omega_o}\,, \\ f(R) & , \quad r_i e^{\omega_o}<R\le R_o\,, \end{cases} \tag{3.10}$$

where

$$f(R)=\frac{R\left(R\alpha'+2\alpha\right)}{R^2+r_i^2\left(1-e^{2\omega_o}\right)}-\frac{R^3(\alpha-\beta)}{\left(R^2+r_i^2\left(1-e^{2\omega_o}\right)\right)^2}+\frac{(2\beta-R\beta')\left(R^2+r_i^2\left(1-e^{2\omega_o}\right)\right)}{R^3}-\frac{\alpha+3\beta}{R}+\beta'. \tag{3.11}$$

Assuming zero traction at the boundary $R=R_0$ we obtain

$$p(R)=\begin{cases} \alpha(R)+\hat{\sigma}_o & , \quad 0\le R\le r_i e^{\omega_o}\,, \\ \displaystyle\int_{R_o}^{R} f(\xi)d\xi+\frac{R_o^2}{R_o^2+r_i^2\left(1-e^{2\omega_o}\right)}\alpha(R_o)+\left(1-\frac{R_o^2+r_i^2\left(1-e^{2\omega_o}\right)}{R_o^2}\right)\beta(R_o) & , \quad r_i e^{\omega_o}<R\le R_o\,, \end{cases} \tag{3.12}$$

where $\hat{\sigma}_o$ is a constant to be determined by enforcing the continuity of the traction vector across the boundary of the ambient metric inclusion, i.e., continuity of σ^{rr} across the disk of radius r_i in the deformed configuration. Following (3.5), the non-zero physical components of the Cauchy stress tensor read

$$\hat{\sigma}^{rr}=\begin{cases} \hat{\sigma}_o & , \quad 0\le R\le r_i e^{\omega_o}\,, \\ \displaystyle\int_{R}^{R_o} f(\xi)d\xi+\frac{R^2}{R^2+r_i^2\left(1-e^{2\omega_o}\right)}\alpha(R)+\left(1-\frac{R^2+r_i^2\left(1-e^{2\omega_o}\right)}{R^2}\right)\beta(R) & \\ \qquad -\displaystyle\frac{R_o^2}{R_o^2+r_i^2\left(1-e^{2\omega_o}\right)}\alpha(R_o)-\left(1-\frac{R_o^2+r_i^2\left(1-e^{2\omega_o}\right)}{R_o^2}\right)\beta(R_o) & , \quad r_i e^{\omega_o}<R\le R_o\,. \end{cases} \tag{3.13}$$

$$\hat{\sigma}^{\theta\theta}=\begin{cases} \hat{\sigma}_o & , \quad 0\le R\le r_i e^{\omega_o}\,, \\ \hat{\sigma}^{rr}+\left[\displaystyle\frac{R^2+r_i^2\left(1-e^{2\omega_o}\right)}{R^2}-\frac{R^2}{R^2+r_i^2\left(1-e^{2\omega_o}\right)}\right]\left(\alpha(R)+\beta(R)\right) & , \quad r_i e^{\omega_o}<R\le R_o\,, \end{cases} \tag{3.14}$$

where $\hat{\sigma}_o$ is a constant given by:

$$\begin{aligned}\hat{\sigma}_o=\int_{r_i e^{\omega_o}}^{R_o} f(\xi)d\xi&+\frac{r_i^2 e^{2\omega_o}}{r_i^2 e^{2\omega_o}+r_i^2\left(1-e^{2\omega_o}\right)}\alpha(r_i e^{\omega_o})+\left(1-\frac{r_i^2 e^{2\omega_o}+r_i^2\left(1-e^{2\omega_o}\right)}{r_i^2 e^{2\omega_o}}\right)\beta(r_i e^{\omega_o}) \\ &-\frac{R_o^2}{R_o^2+r_i^2\left(1-e^{2\omega_o}\right)}\alpha(R_o)-\left(1-\frac{R_o^2+r_i^2\left(1-e^{2\omega_o}\right)}{R_o^2}\right)\beta(R_o)\,.\end{aligned} \tag{3.15}$$

Therefore, for an arbitrary incompressible isotropic solid, the stress is hydrostatic inside the metric inclusion and is equal to $\hat{\sigma}_o$. Note that because of the flat geometry of the ambient space, the second fundamental form of the ambiant plane in $\mathcal{Q}=\mathbb{R}^3$ is equal to zero, and hence the extrinsic equilibrium equations (2.21) yields a zero extrinsic body force b^n on the disk.

Let us assume the particular case of a disk made of a homogeneous neo-Hookean solid, i.e., $\alpha(R) = \mu$ and $\beta(R) = 0$. Therefore, the non-zero physical components of the Cauchy stress (3.13) and (3.14) simplify to:

$$\hat{\sigma}^{rr} = \begin{cases} \hat{\sigma}_o & , \quad 0 \le R \le r_i e^{\omega_o}, \\ \dfrac{\mu}{2}\left[\dfrac{R^2}{R^2 + r_i^2\left(1-e^{2\omega_o}\right)} - \dfrac{R_o^2}{R_o^2 + r_i^2\left(1-e^{2\omega_o}\right)} - \log\left(\dfrac{R^2 + r_i^2\left(1-e^{2\omega_o}\right)}{R_o^2 + r_i^2\left(1-e^{2\omega_o}\right)}\dfrac{R_o^2}{R^2}\right)\right] & , \quad r_i e^{\omega_o} < R \le R_o. \end{cases} \tag{3.16}$$

$$\hat{\sigma}^{\theta\theta} = \begin{cases} \hat{\sigma}_o & , \quad 0 \le R \le r_i e^{\omega_o}, \\ \hat{\sigma}^{rr} + \mu\dfrac{R^2 + r_i^2\left(1-e^{2\omega_o}\right)}{R^2} - \mu\dfrac{R^2}{R^2 + r_i^2\left(1-e^{2\omega_o}\right)} & , \quad r_i e^{\omega_o} < R \le R_o, \end{cases} \tag{3.17}$$

where

$$\hat{\sigma}_o = \frac{\mu}{2}\left[\frac{r_i^2 e^{2\omega_o}}{r_i^2 e^{2\omega_o} + r_i^2\left(1-e^{2\omega_o}\right)} - \frac{R_o^2}{R_o^2 + r_i^2\left(1-e^{2\omega_o}\right)} - \log\left(\frac{r_i^2 e^{2\omega_o} + r_i^2\left(1-e^{2\omega_o}\right)}{R_o^2 + r_i^2\left(1-e^{2\omega_o}\right)}\frac{R_o^2}{r_i^2 e^{2\omega_o}}\right)\right]. \tag{3.18}$$

We plot in Fig. 3.1 the profile of stresses in a disk of initial radius R_o, due to a metric inclusion of radius $r_i = 0.1R_o$ and $\omega_o = 0.05$. Note that since $e^{-\omega_o} < 0.1$, we have $r_i < R_o e^{\omega_o}$, hence, the metric inclusion lies entirely inside the spatial disk of radius $r(R_o) = R_o e^{-\omega_o}$. We observe that the stress is indeed hydrostatic inside the metric inclusion. We also observe a discontinuity of the circumferential stress while the radial stress is continuous as expected following the continuity of the traction vector. Finally, we observe that the stress is asymptotically vanishing as we move away from the inclusion.

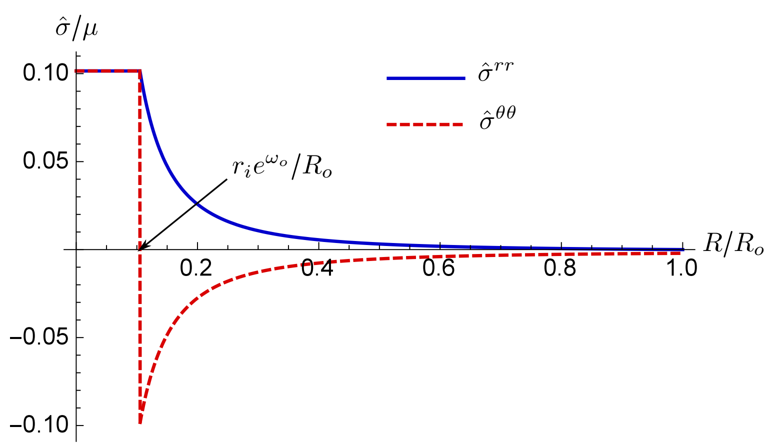

Figure 3.1: *Stresses in a disk of initial radius R_o, due to a metric inclusion of radius $r_i = 0.1R_o$ and $\omega_o = 0.05$.*

Example 3.2. (Spherical cap on a 2D sphere) Let us consider a two-dimensional spherical cap of angular radius Θ_o lying on a sphere of initial radius R_o. We assume that the spherical cap is made of an incompressible and isotropic material. We would like to calculate the stresses that occur in the new equilibrium configuration after the radius of the ambient sphere is changed to r_o. See Fig. 3.2. In spatial spherical coordinates (θ, ϕ), the spatial metric reads

$$\boldsymbol{g} = \begin{pmatrix} r_o^2 & 0 \\ 0 & r_o^2 \sin^2\theta \end{pmatrix}.$$

Note that changing the radius of the sphere from R_o to r_o is equivalent to a uniform scaling of its spatial metric by $e^{2\omega_o} = \frac{r_o^2}{R_o^2}$.

We look for solutions of the form $\varphi(\Theta, \Phi) = (\theta, \phi) = (\theta(\Theta), \Phi)$. Thus, $\boldsymbol{F} = \mathrm{diag}(\theta'(\Theta), 1)$. It follows that the Jacobian is

$$J = \theta'(\Theta)\frac{r_o^2 \sin[\theta(\Theta)]}{R_o^2 \sin\Theta}.$$

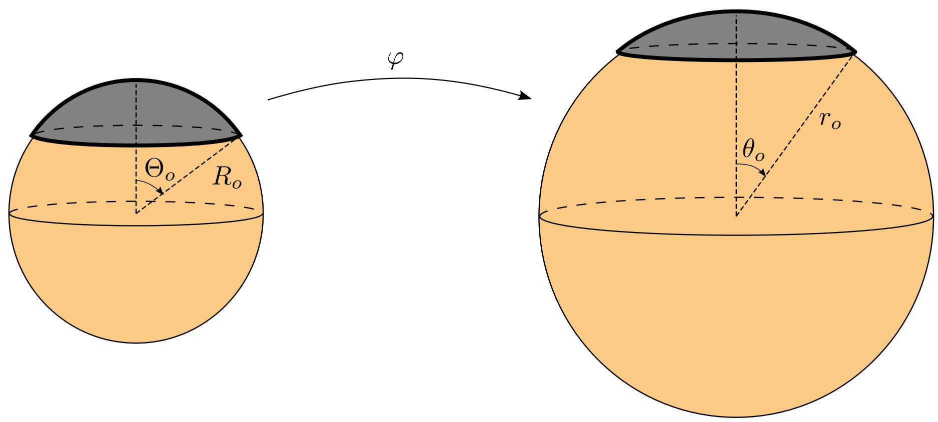

Figure 3.2: *Deformation of a spherical cap by changing the radius of the ambient 2D sphere.*

Assuming that the spherical cap is made of an incompressible material, i.e., $J = 1$, fixing rigid body translations by taking $\theta(0) = 0$, and since $0 \leq \theta < \pi$, we find that

$$\theta(\Theta) = \cos^{-1}\left[\frac{r_o^2}{R_o^2}(\cos\Theta - 1) + 1\right]. \tag{3.19}$$

For example, for $r_o = 1.5R_o$ and Θ_o as we will consider at the end of this section in the numerical example, we see that $\theta_o = \theta(\pi/4) \approx 0.16\pi$. For such a deformation, the Finger tensor reads

$$\boldsymbol{b} = \begin{pmatrix} \frac{R_o^2\sin^2(\Theta)}{r_o^4\sin^2(\theta)} & 0 \\ 0 & \frac{1}{R_o^2\sin^2(\Theta)} \end{pmatrix},$$

and hence $I_1 = \frac{R_o^2\sin^2\Theta}{r_o^2\sin^2\theta} + \frac{r_o^2\sin^2\theta}{R_o^2\sin^2\Theta}$ and $I_2 = 1$. Therefore, we obtain from (3.1) the nonzero stress components as

$$\begin{aligned} \sigma^{\theta\theta} &= -\frac{1}{r_o^2}p + \frac{R_o^2\sin^2\Theta}{r_o^4\sin^2\theta}\alpha + \frac{1}{r_o^2}\left(1 - \frac{r_o^2\sin^2\theta}{R_o^2\sin^2\Theta}\right)\beta, \\ \sigma^{\phi\phi} &= -\frac{1}{r_o^2\sin^2\theta}p + \frac{1}{R_o^2\sin^2\Theta}\alpha + \frac{1}{r_o^2\sin^2\theta}\left(1 - \frac{R_o^2\sin^2\Theta}{r_o^2\sin^2\theta}\right)\beta, \end{aligned} \tag{3.20}$$

where $p(\Theta)$ is the unknown Lagrange multiplier and

$$\alpha(\Theta) = 2\frac{\partial W(I_1, I_2)}{\partial I_1}, \quad \beta(\Theta) = 2\frac{\partial W(I_1, I_2)}{\partial I_2}.$$

By using (3.19), we can write the physical components of stress (3.20) as follows

$$\begin{aligned} \hat{\sigma}^{\theta\theta} &= -p + \frac{r_o^2(\cos\Theta + 1)}{2r_o^2 + R_o^2\cos\Theta - R_o^2}\alpha + \left[1 - \frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{r_o^2(\cos\Theta + 1)}\right]\beta, \\ \hat{\sigma}^{\phi\phi} &= -p + \frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{r_o^2(\cos\Theta + 1)}\alpha + \left[1 - \frac{r_o^2(\cos\Theta + 1)}{2r_o^2 + R_o^2\cos\Theta - R_o^2}\right]\beta. \end{aligned} \tag{3.21}$$

In terms of the Cauchy stress tensor, the only non-trivial intrinsic equilibrium equation is ${\sigma^{\theta a}}_{|a} = 0$, which reads

$$\frac{r_o^2\sin^2\theta}{R_o^2\sin^2\Theta}{\sigma^{\theta\theta}}_{,\Theta} + \frac{1}{\tan\theta}\sigma^{\theta\theta} - \sin\theta\cos\theta\sigma^{\phi\phi} = 0. \tag{3.22}$$

By using (3.19) and (3.20), the equilibrium equation (3.22) reduces to

$$
\begin{aligned}
p' =& -\tan^2\left(\frac{\Theta}{2}\right)\beta' + \frac{r_o^2\alpha'}{R_o^2} - \frac{\tan\left(\frac{\Theta}{2}\right)\left(r_o^2-2R_o^2\right)(\alpha+\beta)}{2r_o^2} + \frac{r_o^2\sin(\Theta)(r_o^2-R_o^2)(\beta-\alpha)}{(2r_o^2+R_o^2\cos\Theta-R_o^2)^2} \\
&+ \frac{R_o^2\tan^2\left(\frac{\Theta}{2}\right)\beta'}{r_o^2} + \frac{8\sin^4\left(\frac{\Theta}{2}\right)\left(R_o^2-r_o^2\right)\beta}{r_o^2\sin^3\Theta} + \frac{4r_o^2\left(R_o^2-r_o^2\right)\alpha' + R_o^2\sin(\Theta)\left(R_o^2-2r_o^2\right)(\alpha+\beta)}{4r_o^2R_o^2+2R_o^4\cos\Theta-2R_o^4}\,.
\end{aligned}
\tag{3.23}
$$

Note that, unlike in the previous example, the spherical cap does not necessarily remain stress-free by a uniform scaling of the spatial metric of the sphere. Note that, if we take $r_o = R_o$, then (3.23) reduce to $p' = \alpha'$, which yields no stress by assuming zero boundary traction at $\Theta = \Theta_o$. Hence, we recover the case of a trivial embedding. The evolving ambient sphere can be isometrically embedded in $\mathbb{R}^3$, i.e., $\mathcal{Q} = \mathbb{R}^3$ where the second fundamental form of the sphere reads

$$
\boldsymbol{k} = \begin{pmatrix} -r_o & 0 \\ 0 & -r_o\sin^2\theta \end{pmatrix}.
$$

We only have one extrinsic equilibrium equation (2.21) which yields the normal component of the body force required to balance the stress field in the spherical cap; it is written as follows

$$
b^n = \frac{1}{\rho}\boldsymbol{\sigma}\!:\!\boldsymbol{k} = -\frac{\hat{\sigma}^{\theta\theta}+\hat{\sigma}^{\phi\phi}}{r_o\rho}\,. \tag{3.24}
$$

In the following, we explore the particular case when the spherical cap is made of a neo-Hookean solid, i.e., $\alpha(R) = \mu$ and $\beta(R) = 0$. Therefore, for a neo-Hookean solid, (3.23) reduces to

$$
p' = \mu\frac{\tan\left(\frac{\Theta}{2}\right)\left(2R_o^2-r_o^2\right)}{2r_o^2} - \mu\frac{r_o^2\sin(\Theta)(r_o^2-R_o^2)}{(2r_o^2+R_o^2\cos(\Theta)-R_o^2)^2} + \mu\frac{\sin(\Theta)\left(R_o^2-2r_o^2\right)}{4r_o^2+2R_o^2\cos(\Theta)-2R_o^2}\,. \tag{3.25}
$$

Therefore, assuming zero boundary traction at $\Theta = \Theta_o$, i.e., $\sigma^{\theta\theta}(\Theta_o) = 0$, we find that

$$
p(\Theta) = \mu\left[g(\Theta) - g(\Theta_o) + \frac{r_o^2(\cos\Theta_o+1)}{2r_o^2+R_o^2\cos\Theta_o-R_o^2}\right], \tag{3.26}
$$

where

$$
g(\Theta) = \frac{2r_o^2-R_o^2}{2R_o^2}\log\left(2r_o^2+R_o^2\cos\Theta-R_o^2\right) + \frac{2R_o^2-r_o^2}{2r_o^2}\log\left[\cos^2\left(\frac{\Theta}{2}\right)\right] - \frac{r_o^2\left(r_o^2-R_o^2\right)}{R_o^2\left(2r_o^2+R_o^2\cos\Theta-R_o^2\right)}\,. \tag{3.27}
$$

Therefore, the stress field (3.21) and the extrinsic body force (3.24) are given by .

$$
\begin{aligned}
\hat{\sigma}^{\theta\theta} =& \mu\Bigg[\frac{R_o^2-2r_o^2}{2R_o^2}\log\left(\frac{2r_o^2+R_o^2\cos\Theta-R_o^2}{2r_o^2+R_o^2\cos\Theta_o-R_o^2}\right) + \frac{r_o^2\cos\Theta}{2r_o^2+R_o^2\cos\Theta-R_o^2} - \frac{r_o^2\cos\Theta_o}{2r_o^2+R_o^2\cos\Theta_o-R_o^2} \\
&+ \frac{r_o^2-2R_o^2}{2r_o^2}\log\left(\frac{1+\cos\Theta_o}{1+\cos\Theta}\right) - \frac{r_o^4(\cos\Theta-\cos\Theta_o)}{(2r_o^2+R_o^2\cos\Theta-R_o^2)\,(2r_o^2+R_o^2\cos\Theta_o-R_o^2)}\Bigg] \\
\hat{\sigma}^{\phi\phi} =& \mu\Bigg[\frac{R_o^2-2r_o^2}{2R_o^2}\log\left(\frac{2r_o^2+R_o^2\cos\Theta-R_o^2}{2r_o^2+R_o^2\cos\Theta_o-R_o^2}\right) + \frac{2r_o^2+R_o^2\cos\Theta-R_o^2}{r_o^2(\cos\Theta+1)} - \frac{r_o^2(\cos\Theta_o+1)}{2r_o^2+R_o^2\cos\Theta_o-R_o^2} \\
&+ \frac{r_o^2-2R_o^2}{2r_o^2}\log\left(\frac{1+\cos\Theta_o}{1+\cos\Theta}\right) - \frac{r_o^2(r_o^2-R_o^2)(\cos\Theta-\cos\Theta_o)}{(2r_o^2+R_o^2\cos\Theta-R_o^2)\,(2r_o^2+R_o^2\cos\Theta_o-R_o^2)}\Bigg] \\
r_o\rho b^n =& \mu\Bigg[\frac{2r_o^2-R_o^2}{R_o^2}\log\left(\frac{2r_o^2+R_o^2\cos\Theta-R_o^2}{2r_o^2+R_o^2\cos\Theta_o-R_o^2}\right) + \frac{r_o^2(\cos\Theta_o+1)}{2r_o^2+R_o^2\cos\Theta_o-R_o^2} - \frac{2r_o^2+R_o^2\cos\Theta-R_o^2}{r_o^2(\cos\Theta+1)} \\
&+ \frac{2R_o^2-r_o^2}{r_o^2}\log\left(\frac{1+\cos\Theta_o}{1+\cos\Theta}\right)\Bigg].
\end{aligned}
\tag{3.28}
$$

We plot in Fig. 3.2 the profile of stresses and the extrinsic body force in a spherical cap of angular radius $\Theta_o = \frac{\pi}{4}$ initially lying on a sphere of radius R_o, due to a change of the radius of the ambient sphere to $r_o = 1.5R_o$.

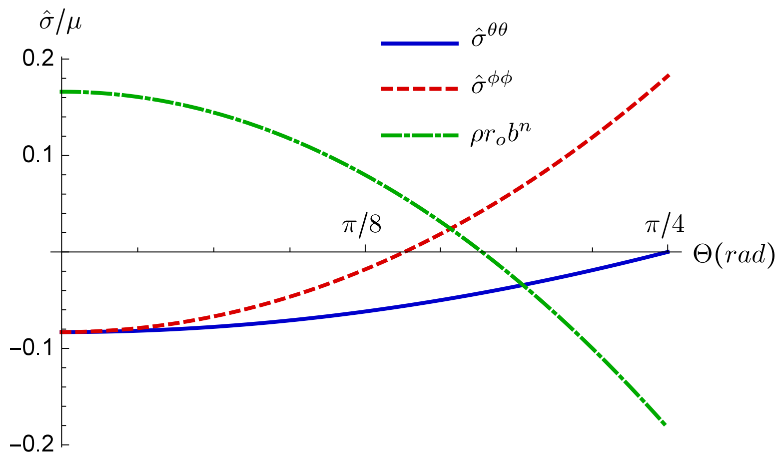

Figure 3.3: *Stresses and the extrinsic body force in a spherical cap of initial angular radius* $\Theta_o = \frac{\pi}{4}$ *initially lying on a sphere of radius* R_o *, due to a change of the radius of the ambient sphere to* $r_o = 1.5R_o$ *.*

We consider the limiting case where $r_o \to \infty$, which corresponds to flattening the spharical cap so that it deforms into the flat Euclidean plane. We obtain from (3.29) that the stress field when the spherical cap is flattened is given by

$$\begin{aligned}\hat{\sigma}^{\theta\theta} &= -\mu\left[\frac{1}{2}\log\left(\frac{1+\cos\Theta}{1+\cos\Theta_o}\right)+\frac{1}{4}(\cos\Theta-\cos\Theta_o)\right]\\ \hat{\sigma}^{\phi\phi} &= -\mu\left[\frac{1}{2}\log\left(\frac{1+\cos\Theta}{1+\cos\Theta_o}\right)+\frac{3}{4}(\cos\Theta-\cos\Theta_o)-\frac{2}{\cos\Theta+1}+\frac{\cos\Theta_o+1}{2}\right].\end{aligned} \tag{3.29}$$

Note that the extrinsic body force field b^n vanishes when $r_o \to \infty$, since this case corresponds to a flat geometry of the ambient space.

3.2 Elastic deformations due to linear perturbations of the ambient space metric

In this section, we linearize the governing equations of the nonlinear theory presented in the previous sections about a reference motion. This will shed light on the mechanical effects of a slight deformation of ambient space on the equilibrium configuration of a deformable body.

Geometric linearization of elasticity was first introduced by Marsden and Hughes [1983] and was further developed by Yavari and Ozakin [2008]. See also Mazzucato and Rachele [2006]. Given a reference motion, we obtain the linearized governing equations with respect to this motion. Suppose a given solid is in a static equilibrium configuration, φ in an ambient space with metric $\boldsymbol{g}$. Let $\boldsymbol{g}_\epsilon$ be a 1-parameter family of spatial metrics, φ_ϵ be the equilibrium configurations, and $\boldsymbol{P}_\epsilon$ be the first Piola-Kirchhoff stresses. Let $\epsilon = 0$ describe the reference motion. Now, for a fixed point $\boldsymbol{X}$ in the material manifold, $\varphi_\epsilon(\boldsymbol{X})$ describes a curve in the spatial manifold, and its derivative at $\epsilon = 0$ gives the variation $\delta\varphi$ as a vector $\boldsymbol{U}(\boldsymbol{X})$ at $\varphi(\boldsymbol{X})$:

$$\delta\varphi(X) = \boldsymbol{U}(\boldsymbol{X}) = \frac{d\varphi_\epsilon(\boldsymbol{X})}{d\epsilon}\Big|_{\epsilon=0}. \tag{3.30}$$

The variation of the ambient space metric is defined as

$$\delta\boldsymbol{g} = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\boldsymbol{g}_\epsilon. \tag{3.31}$$

Now consider, in the absence of body forces, the equilibrium equation $\operatorname{Div}\boldsymbol{P} = \boldsymbol{0}$ for the family of spatial metrics parametrized by ϵ:

$$\operatorname{Div}_\epsilon \boldsymbol{P}_\epsilon = \boldsymbol{0}. \tag{3.32}$$

Linearization of (3.32) is defined as [Marsden and Hughes, 1983; Yavari and Ozakin, 2008]:

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left(\mathrm{Div}_\epsilon\, \boldsymbol{P}_\epsilon\right)=\mathbf{0}. \tag{3.33}$$

Once again, one should note that since the equilibrium configuration is different for each ϵ, $\boldsymbol{P}_\epsilon$ is based at different points in the ambient space for different values of ϵ, and in order to calculate the derivative with respect to ϵ, one in general needs to use the connection (parallel transport) in the ambient space. In components, balance of linear momentum reads

$$\frac{\partial P^{aA}(\epsilon)}{\partial X^A}+\Gamma^A_{AB}P^{aB}(\epsilon)+P^{bB}(\epsilon)\gamma(\epsilon)^a_{bc}F(\epsilon)^c{}_A=0. \tag{3.34}$$

Thus, the linearized balance of linear momentum reads

$$\begin{aligned}\frac{\partial}{\partial X^A}\frac{d}{d\epsilon}\Big|_{\epsilon=0}P^{aA}(\epsilon)+\Gamma^A_{AB}\frac{d}{d\epsilon}\Big|_{\epsilon=0}P^{aB}(\epsilon)+\left[\frac{d}{d\epsilon}\Big|_{\epsilon=0}P^{bB}(\epsilon)\right]\gamma^a_{bc}F^c{}_A\\ +P^{bB}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left[\gamma(\epsilon)^a_{bc}\right]F^c{}_A+P^{bB}\gamma^a_{bc}\,\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left[F(\epsilon)^c{}_A\right]=0.\end{aligned} \tag{3.35}$$

Let us consider a 1-parameter family of metrics $g_{ab}(\epsilon)$ such that

$$g_{ab}(0)=g_{ab},\qquad \frac{d}{d\epsilon}\Big|_{\epsilon=0}g_{ab}(\epsilon)=\delta g_{ab}. \tag{3.36}$$

δg_{ab} is called the metric variation. It can be shown that [Chow et al., 2003]

$$\delta g^{ab}=-g^{am}g^{bn}\delta g_{mn}, \tag{3.37}$$

$$\delta\gamma^a{}_{bc}=\frac{1}{2}g^{ad}\left(\delta g_{cd|b}+\delta g_{bd|c}-\delta g_{bc|d}\right). \tag{3.38}$$

Thus

$$\begin{aligned}\delta\left(\mathrm{Div}\,\boldsymbol{P}\right)^a&=\frac{\partial}{\partial X^A}\delta P^{aA}+\Gamma^A_{AB}\delta P^{aB}+\gamma^a_{bc}F^c{}_A\delta P^{bB}+\delta\gamma^a_{bc}F^c{}_AP^{bB}+\gamma^a_{bc}\delta F^c{}_AP^{bB}\\ &=\delta P^{aA}{}_{|A}+\frac{1}{2}P^{bB}F^c{}_Ag^{ad}\left(\delta g_{cd|b}+\delta g_{bd|c}-\delta g_{bc|d}\right)+P^{bB}\gamma^a_{bc}U^c{}_{|A}.\end{aligned} \tag{3.39}$$

If the initial equilibrium configuration is stress-free, we have

$$\delta P^{aA}{}_{|A}=0. \tag{3.40}$$

Note that

$$P^{aA}=g^{ab}\frac{\partial\Psi}{\partial F^b_A}, \tag{3.41}$$

where $\Psi=\Psi(\boldsymbol{X},\Theta,\boldsymbol{F},\boldsymbol{G},\boldsymbol{g})$ is the material free energy density and Θ is the absolute temperature. In calculating $\frac{dP^{aA}(\epsilon)}{d\epsilon}$, we need to consider the change in $\boldsymbol{F}$ due to the change in the ambient space metric as follows:

$$\frac{dP^{aA(\epsilon)}}{d\epsilon}=\frac{d}{d\epsilon}g^{ab}(\epsilon)\frac{\partial\Psi}{\partial F^b{}_A}+g^{ab}\frac{\partial^2\Psi}{\partial F^c{}_C\partial F^b{}_A}\frac{dF^c{}_C(\epsilon)}{d\epsilon}+g^{ab}\frac{\partial^2\Psi}{\partial g_{cd}\partial F^b{}_A}\frac{dg_{cd}(\epsilon)}{d\epsilon}. \tag{3.42}$$

Let us define

$$\mathbb{A}^a{}_c{}^{CA}=g^{ab}\frac{\partial^2\Psi}{\partial F^c{}_C\partial F^b{}_A}\qquad\text{and}\qquad \mathbb{B}^{acdA}=g^{ab}\frac{\partial^2\Psi}{\partial g_{cd}\partial F^b{}_A}, \tag{3.43}$$

where the derivatives are evaluated at the reference configuration corresponding to $\epsilon=0$. Note that $\mathbb{A}$ is the equivalent of the elasticity tensor in classical linear elasticity. Theretofore

$$\delta P^{aA}{}_{|A}=-g^{ac}P^{dA}\delta g_{cd}+\mathbb{A}^a{}_c{}^{CA}U^c{}_{|C}+\mathbb{B}^{acdA}\delta g_{cd}=\mathbb{A}^a{}_c{}^{CA}U^c{}_{|C}+\mathbb{B}^{acdA}\delta g_{cd}, \tag{3.44}$$

as the initial configuration is assumed to be stress free. With these results, the linearized balance of linear momentum (3.33) is simplified to read

$$\left[\mathbb{A}^{a}{}_{c}{}^{CA} U^{c}{}_{|C} + \mathbb{B}^{acdA} \delta g_{cd}\right]_{|A} = 0 \qquad \text{or} \qquad \operatorname{Div}\left(\mathbb{A} \cdot \nabla \boldsymbol{U} + \mathbb{B} \cdot \delta \boldsymbol{g}\right) = \mathbf{0}. \tag{3.45}$$

Given $\delta \boldsymbol{g}$, the above equations are the governing equations for the displacement field $\boldsymbol{U}$ that results from this change of spatial metric.

We know that for a body deforming quasi-statically in an ambient space with a fixed background metric, linearized balance of linear momentum reads [Yavari and Ozakin, 2008]

$$\left[\mathbb{A}^{a}{}_{c}{}^{CA} U^{c}{}_{|C}\right]_{|A} + \rho_0 (\mathfrak{L} B_0)^{a} = 0, \tag{3.46}$$

where $\mathfrak{L}\boldsymbol{B}$ is the linearized body force. It is seen that in the absence of (mechanical) body forces and when the ambient space is deformed, one can think of $\operatorname{Div}(\mathbb{B} \cdot \delta \boldsymbol{g})$ as an effective body force. In other words, deformation of the ambient space and the equivalent body force will have the same mechanical effect on the deformable body.

Initially Euclidean metric. Let us assume that the initial metric is Euclidean and is isotropically rescaled, i.e. consider a one-parameter family of spatial metrics of the form $(g_\epsilon(\boldsymbol{x}))_{ab} = e^{2\omega_\epsilon(\boldsymbol{x})} \delta_{ab}$. Thus $\delta g_{ab} = 2 \delta\omega \delta_{ab}$. In this case (3.45) reads

$$\left[\mathbb{A}^{a}{}_{c}{}^{CA} U^{c}{}_{,C} + 2 \mathbb{B}^{acdA} \delta_{cd} \delta\omega\right]_{,A} = 0. \tag{3.47}$$

When $\mathbb{A}$ and $\mathbb{B}$ are constants (homogeneous medium), the above equation reads

$$\mathbb{A}^{a}{}_{c}{}^{CA} U^{c}{}_{,CA} + 2 \mathbb{B}^{acdA} \delta_{cd} F^{b}{}_{A} \delta\omega_{,b} = 0. \tag{3.48}$$

Now if $\delta\omega$ is independent of $\boldsymbol{x}$, one finds that $\boldsymbol{U} = \boldsymbol{c}$ (a constant vector) is a solution, i.e. the body will stay stress free in the new (Euclidean) ambient space.

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A Geometry of Riemannian submanifolds

In the following, we tersely review a few elements of the geometry of embedded submanifolds. Here we mainly follow do Carmo [1992]; Capovilla and Guven [1995]; Spivak [1999] and Kuchař [1976]. Let us consider a Riemannian manifold $\mathcal{S}$ embedded in another Riemannian manifold $\mathcal{Q}$ and assume that $\dim \mathcal{S} < \dim \mathcal{Q}$. We consider a time-dependent embedding $\psi_t : \mathcal{S} \to \mathcal{Q}$. The metric $\boldsymbol{h}$ on $\mathcal{Q}$ induces a metric $\boldsymbol{g}_t = \psi_t^* \boldsymbol{h}$ on $\mathcal{S}$ (the first fundamental form). At any given point p of $\mathcal{S}$, the tangent space $T_p\mathcal{S}_t$ has an orthogonal complement $(T_p\mathcal{S}_t)^\perp \subset T\mathcal{Q}$ such that

$$T_p\mathcal{Q} = T_p\mathcal{S}_t \oplus (T_p\mathcal{S}_t)^\perp . \tag{A.1}$$

Note that such a decomposition is smooth in the sense that any smooth vector field $\boldsymbol{u}$ on $\mathcal{S}_t$ can be smoothly decomposed into a vector field $\boldsymbol{u}_\parallel$ tangent to $\mathcal{S}_t$ and a vector field $\boldsymbol{u}_\perp$ normal to $\mathcal{S}_t$, so that $p \to (\boldsymbol{u}_\parallel)_p = (\boldsymbol{u}_p)_\parallel$ and $p \to (\boldsymbol{u}_\perp)_p = (\boldsymbol{u}_p)_\perp$ are smooth. We write $\boldsymbol{u} = \boldsymbol{u}_\parallel + \boldsymbol{u}_\perp$. Let $\dim \mathcal{S} = n$ and $\dim \mathcal{Q} = n + k = m$. Following the smoothness of the decomposition (A.1), we can take a set of smooth vector fields $\{\boldsymbol{\eta}_i\}_{i=1,\dots,k}$ normal to $\mathcal{S}_t$ such that they form an orthonormal basis for $\mathfrak{X}^\perp(\mathcal{S}_t)$, the set of vector fields normal to $\mathcal{S}_t$. Hence, every vector field $\boldsymbol{u}$ on $\mathcal{S}_t$ can be written as $\boldsymbol{u} = \boldsymbol{u}_\parallel + \sum_{i=1}^k u^i \boldsymbol{\eta}_i$. Note that, for $i, j \in \{1, \dots, k\}$, we have $\langle\!\langle \boldsymbol{\eta}_i, \boldsymbol{\eta}_j \rangle\!\rangle_{\boldsymbol{h}} = \delta_{ij}$ and $\langle\!\langle \boldsymbol{\eta}_i, \boldsymbol{u}_\parallel \rangle\!\rangle_{\boldsymbol{h}} = 0$, where the Kronecker delta symbol δ_{ij} is defined as: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

We know that for vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ defined on $\mathcal{S}_t$ and $\mathcal{Q}$, respectively, such that $\boldsymbol{Y}$ is everywhere tangent to $\mathcal{S}_t$, $\nabla^{\boldsymbol{g}_t}_{\psi_t^*\boldsymbol{X}} \psi_t^* \boldsymbol{Y} = \psi_t^*(\nabla^{\boldsymbol{h}}_{\boldsymbol{X}} \boldsymbol{Y})_\parallel$,[15] As a corollary, given a curve c in $\mathcal{S}_t$ and $\boldsymbol{X}$ a vector field along c tangent to $\mathcal{S}_t$ everywhere, $D^{\boldsymbol{g}_t}_s \psi_t^* \boldsymbol{X} = (D^{\boldsymbol{h}}_s \boldsymbol{X})_\parallel$, where $D^{\boldsymbol{g}_t}_s = \nabla^{\boldsymbol{g}_t}_{\frac{\partial}{\partial s}}$. For $i \in \{1, \dots, k\}$, the i^{th} second fundamental form of $\mathcal{S}$ along $\boldsymbol{\eta}_i$ is a $\binom{0}{2}$-tensor $\boldsymbol{k}_{(i)t}$ on $\mathcal{S}$ defined as $\boldsymbol{k}_{(i)t} = \psi_t^* \boldsymbol{\kappa}_{(i)t}$, where [do Carmo, 1992; Capovilla and Guven, 1995]

$$\boldsymbol{\kappa}_{(i)t}(\boldsymbol{u}, \boldsymbol{w}) = \langle\!\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{u}} \boldsymbol{\eta}_i, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{h}} \qquad \forall\ \boldsymbol{u}, \boldsymbol{w} \in T_{\tilde{\boldsymbol{x}}}\mathcal{S}, \tag{A.2}$$

and $\boldsymbol{\eta}_i$, for $i \in \{1, \dots, k\}$, is the i^{th} unit outward normal of $\mathcal{S}_t$ in $\mathcal{Q}$. The orientations of $\boldsymbol{\eta}_i$, for $i \in \{1, \dots, k\}$, are chosen such that the orientation of $\mathcal{S}_t$ and $\mathcal{Q}$ are consistent in the sense that the induced orientation from $\mathcal{S}_t$ along with the ordered sequence $\{\boldsymbol{\eta}_i\}_{i \in \{1,\dots,k\}}$, is equivalent to the orientation of $\mathcal{Q}$. It is known that $\boldsymbol{k}_{(i)t}$ is a symmetirc tensor and can equivalently be written as

$$\boldsymbol{\kappa}_{(i)t} = (\nabla^{\boldsymbol{h}} \boldsymbol{\eta}_i)^\flat , \quad i = 1, \dots, k .$$

For vector fields $\boldsymbol{u}, \boldsymbol{w} \in T_{\boldsymbol{x}}\mathcal{S}$ we can write $\nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \psi_* \boldsymbol{w} = \psi_* \nabla^{\boldsymbol{g}_t}_{\boldsymbol{u}} \boldsymbol{w} + \sum_{i=1}^k b^i(\boldsymbol{u}, \boldsymbol{w}) \boldsymbol{\eta}_i$, where $b^i(.,.)$ is a bilinear form. Therefore, we have

$$b^i(\boldsymbol{u}, \boldsymbol{w}) = \langle\!\langle \nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \psi_* \boldsymbol{w}, \boldsymbol{\eta}_i \rangle\!\rangle_{\boldsymbol{h}} , \quad i = 1, \dots, k .$$

Knowing that $\langle\!\langle \psi_* \boldsymbol{w}, \boldsymbol{\eta}_i \rangle\!\rangle_{\boldsymbol{h}} = 0$ we conclude that

$$\langle\!\langle \nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \psi_* \boldsymbol{w}, \boldsymbol{\eta}_i \rangle\!\rangle_{\boldsymbol{h}} = - \langle\!\langle \nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \boldsymbol{\eta}_i, \psi_* \boldsymbol{w}, \rangle\!\rangle_{\boldsymbol{h}} , \quad i = 1, \dots, k .$$

Hence

$$b^i(\boldsymbol{u}, \boldsymbol{w}) = - \langle\!\langle \nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \boldsymbol{\eta}_i, \psi_* \boldsymbol{w}, \rangle\!\rangle_{\boldsymbol{h}} = -(\nabla^{\boldsymbol{h}} \boldsymbol{\eta}_i)^\flat (\psi_* \boldsymbol{u}, \psi_* \boldsymbol{w}) = -\boldsymbol{k}_{(i)t}(\boldsymbol{u}, \boldsymbol{w}) , \quad i = 1, \dots, k .$$

Therefore, we obtain Gauss's equation

$$\nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \psi_* \boldsymbol{w} = \psi_* \nabla^{\boldsymbol{g}_t}_{\boldsymbol{u}} \boldsymbol{w} - \sum_{i=1}^k \boldsymbol{k}_{(i)t}(\boldsymbol{u}, \boldsymbol{w}) \boldsymbol{\eta}_i .$$

[15] The proof given in [Spivak, 1999] still holds even when the embedding is time dependent. Note that $\nabla^{\boldsymbol{g}_t}$ and $\nabla^{\boldsymbol{h}}$ are the Levi-Civita connections corresponding to $\boldsymbol{g}_t$ and $\boldsymbol{h}$, respectively.

Note that we need to be careful in calculating time derivatives in $(\mathcal{S}, \boldsymbol{g}_t)$, since the induced metric $\boldsymbol{g}_t$ itself depends on time. In particular, when calculating the derivative of the inner product $\langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t}$ of two vector fields $\boldsymbol{u}$ and $\boldsymbol{w}$ along a time-parametrized curve c, the usual formula

$$\frac{d}{dt}\langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} = \langle\!\langle D_t^{\boldsymbol{g}_t}\boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} + \langle\!\langle \boldsymbol{u}, D_t^{\boldsymbol{g}_t}\boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t}, \tag{A.3}$$

is no longer valid when the metric $\boldsymbol{g}_t$ is t-dependent. One instead has[16]

$$\frac{d}{dt}\langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} = \langle\!\langle D_t^{\boldsymbol{g}_t}\boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} + \langle\!\langle \boldsymbol{u}, D_t^{\boldsymbol{g}_t}\boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} + \langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}, \tag{A.4}$$

where

$$\langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} = u^a v^b \frac{\partial g_{t\,ab}}{\partial t}.$$

This can be written in terms of the inner product with respect to $\boldsymbol{g}_t$ as

$$\langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} = \langle\!\langle \boldsymbol{u}, \boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}}{\partial t} \cdot \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t},$$

where $\boldsymbol{g}_t^\sharp$ denotes the "inverse metric", with components g_t^{ab}. Therefore[17]

$$\frac{d}{dt}\langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} = \langle\!\langle D_t^{\boldsymbol{g}_t}\boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} + \langle\!\langle \boldsymbol{u}, D_t^{\boldsymbol{g}_t}\boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} + \langle\!\langle \boldsymbol{u}, \boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t}. \tag{A.6}$$

Using the Levi-Civita connection for the metric $\boldsymbol{g}_t$ to calculate covariant derivatives, the symmetry lemma of classical Riemann geometry [Lee, 1997; Nishikawa, 2002] still holds.[18]

Lemma A.1. *For a Riemannian manifold with a time-dependent metric* $\boldsymbol{g}_t$

$$D_\epsilon^{\boldsymbol{g}_t}\frac{\partial c(t,\epsilon)}{\partial t} = D_t^{\boldsymbol{g}_t}\frac{\partial c(t,\epsilon)}{\partial \epsilon}.$$

The velocity of the time-dependent embedding ψ_t is defined as

$$\boldsymbol{\zeta} = \frac{\partial \psi(t,\boldsymbol{x})}{\partial t} = \boldsymbol{\zeta}_\parallel + \sum_{i=1}^k \zeta^i \boldsymbol{\eta}_i,$$

where $\boldsymbol{\zeta}_\parallel$ is the tangential velocity of the embedding. We also define $\boldsymbol{z} := \psi_t^* \boldsymbol{\zeta}_\parallel$.

Lemma A.1. *For an arbitrary embedding* ψ_t*, the following relation holds*

$$\frac{\partial \boldsymbol{g}_t}{\partial t} = \mathfrak{L}_{\boldsymbol{Z}}\boldsymbol{g}_t + 2\sum_{i=1}^k \zeta^i \boldsymbol{k}_{(i)t}. \tag{A.7}$$

For a transversal embedding, i.e., when $\boldsymbol{\zeta}_\parallel = \boldsymbol{0}$, *(A.7) reduces to*

$$\frac{\partial \boldsymbol{g}_t}{\partial t} = 2\sum_{i=1}^k \zeta^i \boldsymbol{k}_{(i)t}. \tag{A.8}$$

---

[16] Note that $D_t^{\boldsymbol{g}_t}\boldsymbol{g}_t = \frac{\partial \boldsymbol{g}_t}{\partial t}$.

[17] It is also possible to define an alternative covariant time derivative, $\tilde{D}_t^{\boldsymbol{g}_t}$, so that an identity analogous to (A.3) holds. If we let

$$(\tilde{D}_t^{\boldsymbol{g}_t}\boldsymbol{u})^a = \frac{du^a}{dt} + \gamma_{cd}^a u^d \frac{dx^c}{dt} + \frac{1}{2}g^{ab}\frac{\partial g_{bc}}{\partial t}u^c,$$

or in coordinate-free notation

$$\tilde{D}_t^{\boldsymbol{g}_t}\boldsymbol{u} = \nabla_{\boldsymbol{v}}^{\boldsymbol{g}_t}\boldsymbol{u} + \frac{1}{2}\boldsymbol{g}_t^\sharp \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{u}, \tag{A.5}$$

where $\boldsymbol{v} = \dot{c}$, one readily verifies that

$$\frac{d}{dt}\langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} = \langle\!\langle \tilde{D}_t^{\boldsymbol{g}_t}\boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} + \langle\!\langle \boldsymbol{u}, \tilde{D}_t^{\boldsymbol{g}_t}\boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t}.$$

See Thiffeault [2001] for a discussion on this alternative covariant time derivative.

[18] Note that if we were to use the alternative covariant derivative (A.5), this formula would need to be modified.

Proof: First, we note that

$$\boldsymbol{L}_{\boldsymbol{\zeta}}\boldsymbol{h} = \left[\frac{d}{dt}\left(\psi_t \circ \psi_s^{-1}\right)^* \boldsymbol{h}\right]_{s=t} = \left[\frac{d}{dt}\psi_{s*}\psi_t^*\boldsymbol{h}\right]_{s=t} = \left[\frac{d}{dt}\psi_{s*}\boldsymbol{g}_t\right]_{s=t} = \psi_{t*}\left[D_t^{\boldsymbol{g}_t}\boldsymbol{g}_t\right]_{s=t} = \psi_{t*}\frac{\partial \boldsymbol{g}_t}{\partial t}\,. \tag{A.9}$$

On the other hand, we also have

$$\boldsymbol{L}_{\boldsymbol{\zeta}}\boldsymbol{h} = \mathfrak{L}_{\boldsymbol{\zeta}}\boldsymbol{h} = \mathfrak{L}_{\boldsymbol{\zeta}_\parallel}\boldsymbol{h} + \sum_{i=1}^{k}\zeta^i\mathfrak{L}_{\boldsymbol{\eta}_i}\boldsymbol{h}\,.$$

However, for $i \in \{1, \dots, k\}$ we have

$$\left(\mathfrak{L}_{\boldsymbol{\eta}_i}\boldsymbol{h}\right)_{\alpha\beta} = (\eta_i)_{\alpha|\beta} + (\eta_i)_{\beta|\alpha} = 2\kappa_{(i)\alpha\beta}\,. \tag{A.10}$$

We observe that $\mathfrak{L}_{\boldsymbol{\zeta}_\parallel}\boldsymbol{h} = \mathfrak{L}_{\psi_{t*}\boldsymbol{z}}\psi_{t*}\boldsymbol{g}_t$, and, following Marsden and Hughes [1983, p. 98], we have $\mathfrak{L}_{\psi_{t*}\boldsymbol{z}}\psi_{t*}\boldsymbol{g}_t = \psi_{t*}\mathfrak{L}_{\boldsymbol{z}}\boldsymbol{g}_t$. Thus

$$\boldsymbol{L}_{\boldsymbol{\zeta}}\boldsymbol{h} = \psi_{t*}\left(\mathfrak{L}_{\boldsymbol{z}}\boldsymbol{g}_t + 2\sum_{i=1}^{k}\zeta^i\boldsymbol{k}_{(i)t}\right)\,. \tag{A.11}$$

Finally it follows from (A.9) and (A.11) that

$$\frac{\partial \boldsymbol{g}_t}{\partial t} = \mathfrak{L}_{\boldsymbol{z}}\boldsymbol{g}_t + 2\sum_{i=1}^{k}\zeta^i\boldsymbol{k}_{(i)t}\,.$$

■

B An alternative derivation of the tangent balance of linear momentum

In this section, we provide an alternate proof for the tangent balance of linear momentum in the particular case of a transversal evolution of the ambient space. The derivation is based on a generalized version, for arbitrary co-dimension $k = \dim\mathcal{Q} - \dim\mathcal{S}_t$, of a theorem in Marsden and Hughes [1983, p. 129]. The generalized version can be stated as follows (See § 2.1 and Fig. 2.1 to recall the notation):

Theorem B.1. *Assume that given scalar functions a and b, and a vector field $\boldsymbol{c}$ satisfy the following master balance law for any open set $\mathcal{U}$ with C^1 piecewise boundary:*

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} a dv = \int_{\varphi_t(\mathcal{U})} b dv + \int_{\partial\varphi_t(\mathcal{U})}\langle\!\langle \boldsymbol{c}, \mathbf{n}\rangle\!\rangle_{\boldsymbol{g}_t}\, da\,, \tag{B.1}$$

where $\mathbf{n}$ is the unit normal vector to $\partial\varphi_t(\mathcal{U})$ in $\mathcal{S}$. Localization of (B.1) yields

$$\frac{da}{dt} + a\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v} + a\sum_{i=1}^{k}\zeta^i\mathrm{tr}\left(\boldsymbol{k}_{(i)t}\right) = b + \mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{c}\,, \tag{B.2}$$

where we recall that $\boldsymbol{v}$ is the velocity field of φ_t, $\boldsymbol{\zeta} = \sum_{i=1}^{k}\zeta^i\boldsymbol{\eta}_i$ is the velocity field of ψ_t (we have $\boldsymbol{\zeta}_\parallel = \mathbf{0}$ since we are assuming transversal evolution).[19]

Proof: Note that

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} a dv = \frac{d}{dt}\int_{\mathcal{U}} aJdV = \int_{\mathcal{U}}\frac{d}{dt}(aJ)dV = \int_{\mathcal{U}}\left(\frac{da}{dt}J + a\frac{dJ}{dt}\right)dV\,.$$

However

$$\frac{dJ}{dt} = \left(\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v}\right)J + \frac{1}{2}J\,\mathrm{tr}\left(\frac{\partial \boldsymbol{g}_t}{\partial t}\right)\,,$$

[19]Note that what Marsden and Hughes [1983] denote by $\boldsymbol{v}$ is the equivalent of $\boldsymbol{\Upsilon}$ in our notation, so that their $\boldsymbol{v}_\parallel$ corresponds to $\boldsymbol{v}$ (recall that $\boldsymbol{z} = \psi_t^*\boldsymbol{\zeta}_\parallel = \mathbf{0}$) and their v_n would be ζ^n in the particular case when $\mathcal{S}_t$ is a hyperspace of $\mathcal{Q}$.

and following (A.7), we have

$$\frac{\partial \boldsymbol{g}_t}{\partial t} = 2\sum_{i=1}^{k} \zeta^i \boldsymbol{k}_{(i)t}\,.$$

Therefore

$$\begin{aligned}\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} a dv &= \int_{\mathcal{U}} \left(\frac{da}{dt} + a\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v} + a\sum_{i=1}^{k}\zeta^i \mathrm{tr}\left(\boldsymbol{k}_{(i)t}\right)\right) J dV \\ &= \int_{\varphi_t(\mathcal{U})} \left(\frac{da}{dt} + a\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v} + a\sum_{i=1}^{k}\zeta^i \mathrm{tr}\left(\boldsymbol{k}_{(i)t}\right)\right) dv\,.\end{aligned} \tag{B.3}$$

On the other hand, by using Stokes' theorem, we can write

$$\int_{\partial\varphi_t(\mathcal{U})} \langle\!\langle \boldsymbol{c}, \mathbf{n}\rangle\!\rangle_{\boldsymbol{g}_t} da = \int_{\varphi_t(\mathcal{U})} \mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{c}\, dv\,. \tag{B.4}$$

Therefore, by using (B.3) and (B.4), (B.1) transforms to

$$\int_{\varphi_t(\mathcal{U})} \left(\frac{da}{dt} + a\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v} + a\sum_{i=1}^{k}\zeta^i \mathrm{tr}\left(\boldsymbol{k}_{(i)t}\right)\right) dv = \int_{\varphi_t(\mathcal{U})} \left(b + \mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{c}\right) dv\,.$$

Thus, by arbitrariness of the subset $\mathcal{U}$, we find that

$$\frac{da}{dt} + a\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v} + a\sum_{i=1}^{k}\zeta^i \mathrm{tr}\left(\boldsymbol{k}_{(i)t}\right) = b + \mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{c}\,. \tag{B.5}$$

■

First, we derive the localized balance of mass. In integral form, balance of mass reads

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho dv = 0\,.$$

Hence, using the above theorem ($a = \rho$, $b = 0$, and $c = 0$), the balance of mass in localized form reads[20]

$$\frac{d\rho}{dt} + \rho\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v} + \rho\sum_{i=1}^{k}\zeta^i \mathrm{tr}\left(\boldsymbol{k}_{(i)t}\right) = 0\,. \tag{B.6}$$

Next, we look at the balance of linear momentum, which in integral form reads

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho\boldsymbol{v} dv = \int_{\varphi_t(\mathcal{U})} \rho\boldsymbol{b} dv + \int_{\partial\varphi_t(\mathcal{U})} \boldsymbol{\sigma}\cdot\mathbf{n} da\,, \tag{B.7}$$

where $\boldsymbol{b}$ is the body force per unit deformed mass, $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\mathbf{n}$ is the unit normal vector to $\partial\varphi_t(\mathcal{U})$.

Remark B.1. Note that (B.7) makes sense only when the ambient space $\mathcal{S}_t$ is endowed with a linear structure. In a general manifold, integrating a vector field does not make sense. Therefore, the proof shown in this appendix is only valid for linear ambient spaces. However, the resulting localized tangent balance of linear momentum (B.8) still holds in the case of a general manifold as shown in § 2.1 using a Lagrangian field theory, see Eq. (2.18).

In order to use the above theorem, we contract the balance of linear momentum (B.7) with an arbitrary time-independent covariantly constant vector field $\boldsymbol{u}$ tangent to $\varphi_t(\mathcal{U})$, i.e.

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \langle\!\langle \rho\boldsymbol{v}, \boldsymbol{u}\rangle\!\rangle_{\boldsymbol{h}} dv = \int_{\varphi_t(\mathcal{U})} \langle\!\langle \rho\boldsymbol{b}, \boldsymbol{u}\rangle\!\rangle_{\boldsymbol{h}} dv + \int_{\partial\varphi_t(\mathcal{U})} \langle\!\langle \boldsymbol{\sigma}\cdot\mathbf{n}, \boldsymbol{u}\rangle\!\rangle_{\boldsymbol{h}} da\,.$$

[20]Note that (B.6) is equivalent (2.22) since by Lemma A.1, we have $\frac{\partial \boldsymbol{g}_t}{\partial t} = 2\sum_{i=1}^{k}\zeta^i\boldsymbol{k}_{(i)t}$.

We can then use the above theorem for $a = \langle\!\langle \rho\boldsymbol{v}, \boldsymbol{u} \rangle\!\rangle_{\boldsymbol{h}}$, $b = \langle\!\langle \rho\boldsymbol{b}, \boldsymbol{u} \rangle\!\rangle_{\boldsymbol{h}}$, and $\boldsymbol{c} = \boldsymbol{\sigma} \cdot \boldsymbol{u}$. Hence, it follows that

$$\frac{d}{dt} \langle\!\langle \rho\boldsymbol{v}, \boldsymbol{u} \rangle\!\rangle_{\boldsymbol{h}} + \langle\!\langle \rho\boldsymbol{v}, \boldsymbol{u} \rangle\!\rangle_{\boldsymbol{h}} \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{v} + \langle\!\langle \rho\boldsymbol{v}, \boldsymbol{u} \rangle\!\rangle_{\boldsymbol{h}} v_n \operatorname{tr}\boldsymbol{k} = \langle\!\langle \rho\boldsymbol{b}, \boldsymbol{u} \rangle\!\rangle_{\boldsymbol{h}} + \operatorname{div}_{\boldsymbol{g}_t} (\boldsymbol{\sigma} \cdot \boldsymbol{u}) \,.$$

Note that

$$\frac{d}{dt} \langle\!\langle \rho\boldsymbol{v}, \boldsymbol{u} \rangle\!\rangle_{\boldsymbol{h}} = \frac{d\rho}{dt} \langle\!\langle \boldsymbol{v}, \boldsymbol{u} \rangle\!\rangle_{\boldsymbol{h}} + \rho \left\langle\!\!\left\langle D_t^{\boldsymbol{h}} \boldsymbol{v}, \boldsymbol{u} \right\rangle\!\!\right\rangle_{\boldsymbol{h}} \,,$$

where $D_t^{\boldsymbol{h}}$ denotes the time covariant derivative with respect to the metric $\boldsymbol{h}$. Therefore, it follows that

$$\left(\frac{d\rho}{dt} + \rho \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{v} + \rho v_n \operatorname{tr}\boldsymbol{k} \right) \langle\!\langle \boldsymbol{v}, \boldsymbol{u} \rangle\!\rangle_{\boldsymbol{h}} + \rho \left\langle\!\!\left\langle D_t^{\boldsymbol{h}} \boldsymbol{v}, \boldsymbol{u} \right\rangle\!\!\right\rangle_{\boldsymbol{h}} = \langle\!\langle \rho\boldsymbol{b}, \boldsymbol{u} \rangle\!\rangle_{\boldsymbol{h}} + \operatorname{div}_{\boldsymbol{g}_t} (\boldsymbol{\sigma} \cdot \boldsymbol{u}) \,.$$

The first term vanishes following the balance of mass (B.6). Thus, we obtain by arbitrariness of $\boldsymbol{u}$ that

$$\rho \left(D_t^{\boldsymbol{h}} \boldsymbol{v} \right)_{\parallel} = \rho \boldsymbol{b} + \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{\sigma} \,. \tag{B.8}$$

Note that $\left(D_t^{\boldsymbol{h}} \boldsymbol{v} \right)_{\parallel}$ is different from $D_t^{\boldsymbol{g}} \boldsymbol{v}$. In fact, we have proved in Proposition 2.1 that

$$\left(D_t^{\boldsymbol{h}} \boldsymbol{v} \right)_{\parallel} = D_t^{\boldsymbol{g}_t} \boldsymbol{v} + 2 \sum_{i=1}^{k} \zeta^i \boldsymbol{g}^\sharp \cdot \boldsymbol{k}_{(i)t} \cdot \boldsymbol{v} = D_t^{\boldsymbol{g}_t} \boldsymbol{v} + \boldsymbol{g}^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{v} \,. \tag{B.9}$$