On the Balance Laws of Nonlinear Hyperelasticity

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Abstract

It is known that the balance laws of hyperelasticity (Green elasticity), i.e., conservation of mass and balance of linear and angular momenta, can be derived using the first law of thermodynamics by postulating its invariance under superposed rigid body motions of the Euclidean ambient space—the Green-Naghdi-Rivlin theorem. In the case of a non-Euclidean ambient space, covariance of the energy balance—its invariance under arbitrary diffeomorphisms of the ambient space—gives all the balance laws and the Doyle-Ericksen formula—the Marsden-Hughes theorem. It is also known that, by assuming the balance laws, and positing the first and second laws of thermodynamics, the Doyle-Ericksen formula can be derived—the Coleman-Noll procedure. In this paper, we propose a generelization of the Coleman-Noll procedure: we show that the Doyle-Ericksen formula as well as the balance laws of hyperelasticity can be derived using the first and second laws of thermodynamics without assuming any (observer) invariance.

Keywords: Nonlinear elasticity, hyperelasticity, Green elasticity, balance laws, laws of thermodynamics.

Mathematics Subject Classification 74A15 · 74B20

1 Introduction

In nonlinear hyperelasticity, one can derive all the balance laws by starting from an energy balance (the first law of thermodynamics) and postulating its invariance under superposed rigid body motions of the ambient space (observer invariance). This idea is due to Green and Rivlin [1964] in the context of Euclidean ambient spaces (Green-Naghdi-Rivlin theorem). More specifically, Green and Rivlin [1964] postulated the balance of energy and its invariance under superposed translational and rotational motions of the Euclidean ambient space. A different version of this theorem is due to Noll [1963] who thought of the superposed motions passively as time-dependent coordinate charts for the Euclidean ambient space.[1] Effectively, Green and Rivlin [1964] viewed superimposed motions actively, whereas Noll [1963] viewed them passively. The invariance idea was subsequently extended to hyperelasticity (Green elasticity) with Riemannian ambient space manifolds by Hughes and Marsden [1977]; they postulated the invariance of the balance of energy under arbitrary diffeomorphisms of the ambient space—covariance of the energy balance. Hughes and Marsden [1977] showed that covariance of the energy balance gives all the balance laws of hyperelasticity and the Doyle-Ericksen formula [Doyle and Ericksen, 1956][2] (see also [Marsden and Hughes, 1983; Yavari et al., 2006]). In this paper, instead of using the first law of thermodynamics and its covariance, and as an extension to the classical Coleman and Noll [1963] procedure, we show that, one can use the first and the second laws of thermodynamics to derive not only the Doyle-Ericksen formula but also all the balance laws of hyperelasticity.

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[1] Recall that a matrix can be regarded as either a linear transformation (active) or representing a change of basis (passive).

[2] It is worth emphasizing that in both [Green and Rivlin, 1964] and [Hughes and Marsden, 1977] it was assumed that the body is made of a material that has an underlying energy function, i.e., they restricted themselves to hyperelasticity. Similar invariance arguments can be used to derive the balance laws of anelasticity, provided that there exists an underlying energy function, e.g., [Yavari, 2010].

2 Kinematics of Finite Deformations

Consider a hyperelastic solid body B represented by an embedded 3-submanifold $\mathcal{B}$ within the ambient space $\mathcal{S}$.[3] Motion of the body B is represented by a time-parametrized family of maps $\varphi_t : \mathcal{B} \to \mathcal{C}_t \subset \mathcal{S}$, mapping the reference (material) configuration $\mathcal{B}$ of the body to its current (spatial) configuration $\mathcal{C}_t = \varphi_t(\mathcal{B})$. We adopt the following standard convention: objects and indices are denoted by uppercase characters in the material manifold $\mathcal{B}$ (e.g., $X \in \mathcal{B}$), and by lowercase characters in the spatial manifold $\mathcal{S}$ (e.g., $x = \varphi_t(X) \in \varphi_t(\mathcal{B})$). We consider local coordinate charts on $\mathcal{B}$ and $\mathcal{S}$ denoted by $\{X^A\}$ and $\{x^a\}$, respectively. The corresponding local coordinate bases are denoted by $\{\partial_A = \frac{\partial}{\partial X^A}\}$ and $\{\partial_a = \frac{\partial}{\partial x^a}\}$, and their respective dual bases are $\{dX^A\}$ and $\{dx^a\}$. We also adopt Einstein's repeated index summation convention, e.g., $u^i v_i := \sum_i u^i v_i$.

The ambient space has a background metric $\mathbf{g} = \mathrm{g}_{ab}\, dx^a \otimes dx^b$. Given vectors $\mathbf{u}, \mathbf{w} \in T_x\mathcal{S}$, their dot product is denoted by $\langle\!\langle \mathbf{u}, \mathbf{w} \rangle\!\rangle_{\mathbf{g}} = \mathrm{u}^a\, \mathrm{w}^b\, \mathrm{g}_{ab}$. Given a vector $\mathbf{u} \in T_x\mathcal{S}$ and a 1-form $\boldsymbol{\omega} \in T_x^*\mathcal{S}$, their natural pairing is denoted by $\langle \boldsymbol{\omega}, \mathbf{u} \rangle = \boldsymbol{\omega}(\mathbf{u}) = \omega_a\, \mathrm{u}^a$. The spatial volume form is $dv = \sqrt{\det \mathbf{g}}\; dx^1 \wedge dx^2 \wedge dx^3$. The Levi-Civita connection of $(\mathcal{S}, \mathbf{g})$ is denoted by $\nabla^{\mathbf{g}}$, with Christoffel symbols $\gamma^a{}_{bc}$. The metric $\mathbf{g}$ of $\mathcal{S}$ induces the metric $\mathbf{G}$ on $\mathcal{B}$ by which the natural distances in the body before deformation are calculated. Given vectors $\mathbf{U}, \mathbf{W} \in T_X\mathcal{B}$, their dot product is denoted by $\langle\!\langle \mathbf{U}, \mathbf{W} \rangle\!\rangle_{\mathbf{G}} = \mathrm{U}^A\, \mathrm{W}^B\, \mathrm{G}_{AB}$. Given a vector $\mathbf{U} \in T_X\mathcal{B}$ and a 1-form $\boldsymbol{\Omega} \in T_X^*\mathcal{B}$, their natural pairing is denoted by $\langle \boldsymbol{\Omega}, \mathbf{U} \rangle = \boldsymbol{\Omega}(\mathbf{U}) = \Omega_A\, \mathrm{U}^A$. The material volume form is $dV = \sqrt{\det \mathbf{G}}\; dX^1 \wedge dX^2 \wedge dX^3$. The Levi-Civita connection of $(\mathcal{B}, \mathbf{G})$ is denoted by $\nabla^{\mathbf{G}}$, with Christoffel symbols $\Gamma^A{}_{BC}$.

As a measure of strain in elastic solids, we typically use the derivative of the deformation mapping—known as the deformation gradient—denoted by $\mathbf{F}(X,t) = T\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{C}_t$; in components it reads as $\mathrm{F}^a{}_A = \frac{\partial \varphi^a}{\partial X^A}$. The dual $\mathbf{F}^\star$ of $\mathbf{F}$ is defined as $\mathbf{F}^\star(X,t) : T_{\varphi_t(X)}\mathcal{C}_t \to T_X\mathcal{B}$, $\langle \boldsymbol{\alpha}, \mathbf{F}\mathbf{U} \rangle = \langle \mathbf{F}^\star\boldsymbol{\alpha}, \mathbf{U} \rangle$, $\forall\, \mathbf{U} \in T_X\mathcal{B}$, $\forall\, \boldsymbol{\alpha} \in T^*_{\varphi(X)}\mathcal{S}$; it has components $(\mathrm{F}^\star)_A{}^a = \mathrm{F}^a{}_A$. The transpose $\mathbf{F}^\mathsf{T}$ of $\mathbf{F}$ is defined as $\mathbf{F}^\mathsf{T}(X,t) : T_{\varphi_t(X)}\mathcal{C}_t \to T_X\mathcal{B}$, $\langle\!\langle \mathbf{F}\mathbf{U}, \mathbf{u} \rangle\!\rangle_{\mathbf{g}} = \langle\!\langle \mathbf{U}, \mathbf{F}^\mathsf{T}\mathbf{u} \rangle\!\rangle_{\mathbf{G}}$, $\forall\, \mathbf{U} \in T_X\mathcal{B}$, $\forall\, \mathbf{u} \in T_{\varphi(X)}\mathcal{S}$; in components it reads as $\big(\mathrm{F}^\mathsf{T}\big)^A{}_a = \mathrm{G}^{AB}\, \mathrm{F}^b{}_B\, \mathrm{g}_{ba}$. Note that $\mathbf{F}^\mathsf{T} = \mathbf{G}^\sharp \mathbf{F}^\star \mathbf{g}$, where $(.)^\sharp$ denotes the musical isomorphism for raising indices. The right Cauchy–Green deformation tensor is defined as $\mathbf{C} := \mathbf{F}^\mathsf{T}\mathbf{F}$. Denoting by $(.)^\flat$ the musical isomorphism for lowering indices, one finds that $\mathbf{C}^\flat$ corresponds to the pull-back of the spatial metric $\mathbf{g}$ by φ, i.e., $\mathbf{C}^\flat = \varphi^*\mathbf{g} = \mathbf{F}^\star\mathbf{g}\mathbf{F}$. The Jacobian of the motion relates the material and spatial volume elements as $\mathrm{d}v = J\mathrm{d}V$, and it can be shown that $J = \sqrt{\det \mathbf{C}} = \sqrt{\det \mathbf{g}/\det \mathbf{G}}\, \det \mathbf{F}$.

The material velocity of the motion is defined as $\mathbf{V} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{V}(X,t) := \frac{\partial \varphi(X,t)}{\partial t}$; it has components $\mathrm{V}^a = \frac{\partial \varphi^a}{\partial t}$. The spatial velocity is defined as $\mathbf{v} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{v}(x,t) := \mathbf{V}(\varphi_t^{-1}(x), t)$. The material acceleration of the motion is defined as $\mathbf{A} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{A}(X,t) := D_t^{\mathbf{g}}\mathbf{V}(X,t)$, where $D_t^{\mathbf{g}}$ denotes the covariant derivative along $\varphi_X : t \mapsto \varphi(X,t)$; it reads in components as $\mathrm{A}^a = \frac{\partial \mathrm{V}^a}{\partial t} + \gamma^a{}_{bc}\, \mathrm{V}^b\, \mathrm{V}^c$. The spatial acceleration of the motion is defined as $\mathbf{a} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S}$, $\mathbf{a}(x,t) := \mathbf{A}(\varphi_t^{-1}(x), t) \in T_x\mathcal{S}$; it has components $\mathrm{a}^a = \frac{\partial \mathrm{v}^a}{\partial t} + \frac{\partial \mathrm{v}^a}{\partial x^b}\, \mathrm{v}^b + \gamma^a{}_{bc}\, \mathrm{v}^b\, \mathrm{v}^c$.

3 Thermodynamics and the Balance Laws of Hyperelasticity

In this section, we first briefly review the first and the second laws of thermodynamics in the setting of nonlinear hyperelasticity. We then show how all the balance laws and constitutive equations of hyperelasticity can be derived departing from the first and the second laws of thermodynamics without assuming any invariance.

Remark 3.1 (Hyperelasticity)**.** In Cauchy elasticity, the stress at any given point is a function of the strain at that same point [Cauchy, 1828; Truesdell, 1952] without any history dependence. However, it is important to note that not all Cauchy elastic solids possess an energy function. Those Cauchy elastic solids that do possess energy functions are often referred to as Green elastic [Green, 1838, 1839; Spencer, 2015] or hyperelastic [Truesdell, 1952]. It is worth noting that not all elastic solids are Cauchy elastic. There are indeed elastic solids whose constitutive equations are implicit, such as those expressed as $\mathbf{f}(\boldsymbol{\sigma}, \mathbf{b}) = \mathbf{0}$ [Morgan, 1966; Rajagopal, 2003, 2007, 2011]. Cauchy elastic solids are a proper subset of this class of solids; and Green elastic solids are a proper subset of Cauchy elastic solids.

[3] For most applications the ambient space is the three-dimensional Euclidean space, i.e., $\mathcal{S} = \mathbb{R}^3$. However, in general, the ambient space may be curved, e.g., in modeling the dynamics of fluid membranes [Arroyo and DeSimone, 2009]. See [Yavari et al., 2016] for a general framework on elasticity in evolving ambient spaces.

3.1 First Law of Thermodynamics

The first law of thermodynamics posits the existence of an internal energy $\mathscr{E}$ as a state function, which satisfies the following balance equation as an expression of the conservation of energy principle [Truesdell, 1952; Gurtin, 1974; Marsden and Hughes, 1983]

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(\mathscr{E} + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right) dV = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \mathbf{B}, \mathbf{V}\rangle\!\rangle_{\mathbf{g}} + R\right) dV + \int_{\partial\mathcal{U}} \left(\langle\!\langle \mathbf{T}, \mathbf{V}\rangle\!\rangle_{\mathbf{g}} + H\right) dA\,, \tag{3.1}$$

for any open set $\mathcal{U} \subset \mathcal{B}$, where $\mathscr{E}$ stands for the specific internal energy, ρ_0 is the material mass density, $\mathbf{B}$ is the specific body force, $\mathbf{T}$ is the boundary traction vector field per unit material area, $R = R(X,t)$ is the specific heat supply, and $H = -\langle\!\langle \mathbf{Q}, \mathbf{N}\rangle\!\rangle_{\mathbf{G}}$ is the heat flux across a material surface where $\mathbf{Q} = \mathbf{Q}(X, \Theta, d\Theta, \mathbf{C}^\flat, \mathbf{G})$ represents the external heat flux per unit material area, Θ is temperature, $d\Theta$ is its exterior derivative, and $\mathbf{N}$ is the $\mathbf{G}$-unit normal to the boundary $\partial\mathcal{U}$. Cauchy's stress theorem tells us that there exists a stress tensor $\mathbf{P}$—the first Piola-Kirchhoff stress tensor—such that $\mathbf{T} = \mathbf{P}\mathbf{N}^\flat$, or in components, $T^a = P^{aA}\,N_A$, where $N_A = G_{AB}\,N^B$. Expressed in localized form, the energy balance (3.1) yields

$$\rho_0\,\dot{\mathscr{E}} = \mathbf{S} : (\overset{s}{\mathbf{D}} + \overset{a}{\mathbf{D}}) - \operatorname{Div}\mathbf{Q} + \rho_0 R + \langle\!\langle \operatorname{Div}\mathbf{P} + \rho(\mathbf{B} - \mathbf{A}), \mathbf{V}\rangle\!\rangle_{\mathbf{g}} - \dot{\rho}_0 \left(\mathscr{E} + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right). \tag{3.2}$$

In the local form of the balance of energy (3.2) a dotted quantity denotes its total time derivative, $\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$ is the second Piola-Kirchhoff stress tensor, and the relation $\mathbf{P} : \nabla\mathbf{V} = \mathbf{S} : \overset{s}{\mathbf{D}} + \mathbf{S} : \overset{a}{\mathbf{D}}$ has been used, where

$$\overset{s}{\mathbf{D}} = \frac{1}{2}\left[\mathbf{F}^\star(\nabla\mathbf{V})\mathbf{F} + \mathbf{F}^\star(\nabla\mathbf{V})^\star\,\mathbf{F}\right] = \frac{1}{2}\dot{\mathbf{C}}^\flat\,, \qquad \overset{a}{\mathbf{D}} = \frac{1}{2}\left[\mathbf{F}^\star(\nabla\mathbf{V})\mathbf{F} - \mathbf{F}^\star(\nabla\mathbf{V})^\star\,\mathbf{F}\right]. \tag{3.3}$$

$\overset{s}{\mathbf{D}} = \frac{1}{2}\dot{\mathbf{C}}^\flat$ is the symmetric material rate of deformation tensor and $\overset{a}{\mathbf{D}}$ is an anti-symmetric tensor. It should be noted that we do not assume the symmetry of $\mathbf{S}$.[4]

3.2 Second Law of Thermodynamics

The second law of thermodynamics posits the existence of specific entropy $\mathscr{N}$ as a state function, which satisfies the following inequality—known as the Clausius-Duhem inequality—as an expression of the principle of entropy production,[5] which steadily increases or remains constant within a closed system over time [Truesdell, 1952; Gurtin, 1974; Marsden and Hughes, 1983]

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \mathscr{N}\, dV \geq \int_{\mathcal{U}} \rho_0 \frac{R}{\Theta} dV + \int_{\partial\mathcal{U}} \frac{H}{\Theta} dA\,, \tag{3.4}$$

for any open set $\mathcal{U} \subset \mathcal{B}$. In localized form, the material Clausius-Duhem inequality (3.4) yields

$$\dot{\eta} = \rho_0\Theta\dot{\mathscr{N}} + \dot{\rho}_0\Theta\mathscr{N} + \operatorname{Div}\mathbf{Q} - \rho_0 R - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0\,, \tag{3.5}$$

where $\dot{\eta}$ denotes the material rate of energy dissipation density, and $d\Theta$ is the exterior derivative of Θ (in components, $\langle d\Theta, \mathbf{Q}\rangle = \frac{\partial\Theta}{\partial X^A}\,Q^A$).

Remark 3.2. At a material point $X \in \mathcal{B}$, Coleman and Noll [1963] called the set

$$\left\{\varphi_t(X), \mathbf{P}(X,t), \mathbf{B}(X,t), \mathscr{E}(X,t), \mathbf{Q}(X,t), R(X,t), \mathscr{N}(X,t), \Theta(X,t)\right\}, \tag{3.6}$$

a *thermodynamic process* when all its eight fields satisfy the first law of thermodynamics and the balance of linear and angular momenta.[6] A given hyperelastic material is specified by its constitutive equations, e.g.,

[4] We thank Sanjay Govindjee for bringing this to our attention.

[5] The entropy production for an open subset $\mathcal{U}$ in the body reads as

$$\Gamma = \frac{d}{dt}\int_{\mathcal{U}} \rho_0 \mathscr{N}\, dV - \int_{\mathcal{U}} \rho_0 \frac{R}{\Theta} dV - \int_{\partial\mathcal{U}} \frac{H}{\Theta}\, dA\,.$$

[6] Instead of the first Piola-Kirchhoff stress, they used the Cauchy stress and assumed its symmetry—the balance of angular momentum.

$\mathscr{E} = \hat{\mathscr{E}}(X, \mathscr{N}, \mathbf{F}, \mathbf{G}, \mathbf{g})$ and $\mathbf{Q} = \hat{\mathbf{Q}}(X, \Theta, d\Theta, \mathbf{F}, \mathbf{G}, \mathbf{g})$. A thermodynamic process is *admissible* if the constitutive equations hold everywhere in the body and at all times. Coleman and Noll [1963] showed that requiring the Clausius-Duhem inequality (3.5) to hold for all admissible thermodynamic processes puts certain constraints on the constitutive assumptions, e.g., the Doyle-Ericksen formula.

At a material point $X \in \mathcal{B}$, we define an *extended thermodynamic process* to be the set (3.6) when all its eight fields satisfy the first law of thermodynamics. An extended thermodynamic process is *admissible* if the constitutive equations hold everywhere in the body and at all times. We require that the Clausius-Duhem inequality (3.5) holds for all admissible extended thermodynamic processes.

3.3 Balance Laws of Nonlinear Hyperelasticity

Let us consider a body $\mathcal{B}$ made of a hyperelastic material at every material point $X \in \mathcal{B}$ for which there exists an energy function $\mathscr{E}$ that depends explicitly on strain at the same material point $X \in \mathcal{B}$, i.e., $\mathscr{E} = \hat{\mathscr{E}}(X, \mathscr{N}, \mathbf{C}^\flat, \mathbf{G})$ [Truesdell, 1952]. The specific free energy may hence be defined as the Legendre transform of the specific internal energy $\mathscr{E}$ with respect to the conjugate variables temperature Θ and specific entropy $\mathscr{N}$, i.e., $\Psi = \hat{\Psi}(X, \Theta, \mathbf{C}^\flat, \mathbf{G})$ and

$$\Psi = \mathscr{E} - \Theta\mathscr{N}\,, \tag{3.7}$$

such that $\mathscr{E} = \hat{\mathscr{E}}(X, \mathscr{N}, \mathbf{C}^\flat, \mathbf{G})$. Thus[7]

$$\mathscr{N} = -\frac{\partial\Psi}{\partial\Theta}\,. \tag{3.8}$$

Proposition 3.1. *For a hyperelastic body,*[8] *the first and second laws of thermodynamics* (3.2) *and* (3.5) *imply that*

$$\begin{cases} \mathbf{P} = 2\rho_0\mathbf{F}\dfrac{\partial\hat{\Psi}}{\partial\mathbf{C}^\flat}\,, & (3.9)\\ \operatorname{Div}\mathbf{P} + \rho_0\mathbf{B} = \rho_0\mathbf{A}\,, & (3.10)\\ \dot{\rho}_0 = 0\,, & (3.11)\\ \dot{\eta} = -\dfrac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0\,. & (3.12) \end{cases}$$

In other words, the first and second laws of thermodynamics imply the Doyle-Ericksen formula (3.9) *(and consequently the balance of angular momentum*[9]*), the balance of linear momentum* (3.10)*, and the conservation of mass* (3.11)*.*

Proof. From (3.7), $\rho_0\Theta\dot{\mathscr{N}} = \rho_0\dot{\mathscr{E}} - \rho_0\dot{\Psi} - \rho_0\dot{\Theta}\mathscr{N}$. Substituting this relation and (3.2) into (3.5) one obtains

$$\dot{\eta} = \mathbf{S}:(\overset{s}{\mathbf{D}} + \overset{a}{\mathbf{D}}) - \rho_0\dot{\Psi} - \rho_0\dot{\Theta}\mathscr{N} + \dot{\rho}_0\Theta\mathscr{N} + \langle\!\langle \operatorname{Div}\mathbf{P} + \rho(\mathbf{B}-\mathbf{A}), \mathbf{V}\rangle\!\rangle_{\mathbf{g}} - \dot{\rho}_0\left[\mathscr{E} + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right] - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0\,. \tag{3.13}$$

Note that

$$\dot{\Psi} = \frac{\partial\Psi}{\partial\Theta}\dot{\Theta} + \frac{\partial\Psi}{\partial\mathbf{C}^\flat}:\dot{\mathbf{C}}^\flat = -\mathscr{N}\dot{\Theta} + 2\frac{\partial\Psi}{\partial\mathbf{C}^\flat}:\overset{s}{\mathbf{D}}\,. \tag{3.14}$$

Substituting this into (3.13), the rate of dissipation is simplified to read

$$\dot{\eta} = \left[\mathbf{S} - 2\rho_0\frac{\partial\Psi}{\partial\mathbf{C}^\flat}\right]:\overset{s}{\mathbf{D}} + \mathbf{S}:\overset{a}{\mathbf{D}} + \langle\!\langle \operatorname{Div}\mathbf{P} + \rho(\mathbf{B}-\mathbf{A}), \mathbf{V}\rangle\!\rangle_{\mathbf{g}} - \dot{\rho}_0\left[\Psi + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right] - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0\,. \tag{3.15}$$

[7] Note that the Legendre transform (3.7) of $\mathscr{E}$ to Ψ with respect to the conjugate variables $\mathscr{N}$ and Θ is essentially a change of variable, from the former to the latter, satisfying (3.8). See [Arnold, 1989; Goldstein et al., 2002] for further details on Legendre transform in the context of Lagrangian mechanics and thermodynamics.

[8] It should be emphasized that this result can be easily extended to hyper-anelasticity, where there is an energy function that explicitly depends on elastic distortions.

[9] Using the symmetry of the right Cauchy-Green deformation tensor (i.e., $\mathbf{C}^\star = \mathbf{C}$) in the Doyle-Ericksen formula yields the balance of linear momentum $\mathbf{P}^\star\mathbf{F}^{-\star} = \mathbf{F}^{-1}\mathbf{P}$, which is equivalent to $\mathbf{F}\mathbf{P}^\star = \mathbf{P}\mathbf{F}^\star$, i.e., symmetry of the Cauchy stress.

This inequality must hold for all motions, i.e., all extended thermodynamic processes. As $\overset{s}{\mathbf{D}}$ (a symmetric tensor) and $\overset{a}{\mathbf{D}}$ (an antisymmetric tensor) can be chosen independently of all the other fields, one concludes that[10]

$$\mathbf{S} = 2\rho_0 \frac{\partial \hat{\Psi}}{\partial \mathbf{C}^\flat}\,, \qquad \mathbf{S}^\star = \mathbf{S}\,. \tag{3.16}$$

Note that as soon as the first relation (the Doyle-Ericksen formula) holds, the second relation (balance of angular momentum) is trivially satisfied by virtue of the symmetry of the right Cauchy-Green deformation tensor. Now the rate of dissipation (3.15) is simplified to read

$$\dot{\eta} = \langle\!\langle \operatorname{Div}\mathbf{P} + \rho(\mathbf{B}-\mathbf{A}), \mathbf{V}\rangle\!\rangle_{\mathbf{g}} - \dot{\rho}_0\left[\Psi + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right] - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0\,. \tag{3.17}$$

One can choose the velocity vector arbitrarily while its norm $\|\mathbf{V}\|_{\mathbf{g}}$ is fixed. This implies that the inequality (3.17) can hold only if

$$\operatorname{Div}\mathbf{P} + \rho_0\mathbf{B} = \rho_0\mathbf{A}\,. \tag{3.18}$$

Now the rate of dissipation takes the following form

$$\dot{\eta} = -\dot{\rho}_0\left[\Psi + \frac{1}{2}\|\mathbf{V}\|^2_{\mathbf{g}}\right] - \frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0\,. \tag{3.19}$$

Next one can choose the velocity vector norm $\|\mathbf{V}\|_{\mathbf{g}}$ arbitrarily while the other fields remain fixed. For all these extended thermodynamics processes the above inequality must hold. This implies that $\dot{\rho}_0 = 0$ and $\dot{\eta} = -\frac{1}{\Theta}\langle d\Theta, \mathbf{Q}\rangle \geq 0$. □

Remark 3.3. Coleman and Noll [1963] showed that requiring (3.5) to hold for all admissible thermodynamic processes gives the Doyle-Ericksen formula (3.9). We have shown that requiring (3.5) to hold for all admissible extended thermodynamic processes gives all the balance laws and the Doyle-Ericksen formula (3.9).

Remark 3.4. For an incompressible hyperelastic solid, the Legendre transform (3.7) is modified to take into account the constraint of volume preservation $J = 1$ on motions as follows

$$\Psi - p(J-1) = \mathscr{E} - \Theta\mathscr{N}\,, \tag{3.20}$$

where $p(X,t)$ is the Lagrange multiplier associated with the incompressibiity constraint. Consequently, since $\dot{J} = \frac{1}{2}J\mathbf{C}^{-\sharp}\!:\!\dot{\mathbf{C}}^\flat$, the Doyle-Ericksen formula (3.9) is modified to read

$$\mathbf{P} = 2\rho_0\mathbf{F}\frac{\partial\hat{\Psi}}{\partial\mathbf{C}^\flat} - p\,\mathbf{g}^\sharp\mathbf{F}^{-\star}\,. \tag{3.21}$$

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[10] There are infinitely may extended thermodynamic processes for which everything except for $\overset{s}{\mathbf{D}}$ and $\overset{a}{\mathbf{D}}$ are the same. For the second law to hold for all such processes one must have (3.16).

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